33. R. W. Hamming, "The Unreasonable Effectiveness of Mathematics", *The American Mathematical Monthly* **87** (1980) 81–90.
34. I. Grattan-Guinness, "Solving Wigner's mystery: The reasonable (though perhaps limited) effectiveness of mathematics in the natural sciences", The Mathematical Intelligencer (2008) 7–17.
35. D. Abbott, "The Reasonable Ineffectiveness of Mathematics", *Proceedings of the IEEE* **101** (2013) 2147–2153.
36. D. Hand, *The Improbability Principle; Why Coincidences, Miracles, and Rare Events Happen Every Day*, Scientific American / Farrar, Straus and Giroux (2014).
37. G. Pólya, *Mathematics and Plausible Reasoning*, Princeton University Press (1954; 2^{nd} edition 1968).
38. H. Putnam, "What is mathematical truth?", *Historia Mathematica* **2** (1975) 529–533.
39. D. Howard, "Einstein and the Development of Twentieth-Century Philosophy of Science", Chapter 11 in: M. Janssen and C. Lehner (eds), The Cambridge Companion to Einstein, Cambridge University Press (2014) pp. 354–376.
40. A. Einstein, "Physics and reality", translated by J. Piccard, The Journal of the Franklin Institute 221 (1936) 3349–382.
41. J.O. Urmson and G.J. Warnock (eds), *J. L. Austin; Philosophical Papers*, Oxford University Press (1979) p. 252.
42. FXQi discussion forum for the current essay, URL: http://fqxi.org/community/forum/topic/2492 (retrieved Sept. 2015).

Mathematics Is Physics

M. S. Leifer

Abstract In this essay, I argue that mathematics is a natural science—just like physics, chemistry, or biology—and that this can explain the alleged "unreasonable" effectiveness of mathematics in the physical sciences. The main challenge for this view is to explain how mathematical theories can become increasingly abstract and develop their own internal structure, whilst still maintaining an appropriate empirical tether that can explain their later use in physics. In order to address this, I offer a theory of mathematical theory-building based on the idea that human knowledge has the structure of a scale-free network and that abstract mathematical theories arise from a repeated process of replacing strong analogies with new hubs in this network. This allows mathematics to be seen as the study of regularities, within regularities, within..., within regularities of the natural world. Since mathematical theories are derived from the natural world, albeit at a much higher level of abstraction than most other scientific theories, it should come as no surprise that they so often show up in physics. This version of the essay contains an addendum responding to Slyvia Wenmackers' essay [1] and comments that were made on the FQXi website [2].

The Unreasonable Effectiveness of Mathematics in the Physical Sciences

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve—Eugene Wigner [3].

Mathematics is the language of physics, and not just any mathematics at that. Our fundamental laws of physics are formulated in terms of some of the most advanced and abstract branches of mathematics. Seemingly abstruse ideas like differential geometry, fibre bundles, and group representations have been commonplace in physics for decades, and theoretical physics is only getting more abstract, with fields like

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category theory playing an increasingly important role in our theory building. Do we simply have to accept this mathematization as a brute fact about our universe, or can it be explained?

My thesis is that mathematics is a natural science—just like physics, chemistry, or biology—albeit one that is separated from empirical data by several layers of abstraction. It is nevertheless fundamentally a theory about our physical universe and, as such, it should come as no surprise that our fundamental theories of the universe are formulated in terms of mathematics.

The philosophical worldview underlying my arguments is that of naturalism [4, 5]. Naturalism is the position that everything arises from natural properties and causes, i.e. supernatural or spiritual explanations are excluded. In particular, natural science is our best guide to what exists, so natural science should guide our theorizing about the nature of mathematical objects.

Mathematics is not usually thought of as an empirical science, so naturalism may seem irrelevant to its philosophy. However, we have one rather conspicuous empirical data point about it, namely the alleged "unreasonable" effectiveness of mathematics in the physical sciences. Since our fundamental laws of physics are formulated in terms of some of the most advanced branches of mathematics, a philosophy of mathematics that explains this should be preferred to one in which it is an "unreasonable" accident or miracle. Such a theory would also be falsifiable in the sense that, were it to be the case our future fundamental theories of physics resist mathematization, being only explainable in words or only formalizable in terms of very elementary mathematics, then our philosophy of mathematics would have been proved wrong.

The philosophy of mathematics I develop here is closely related to those of Quine and Putnam [6–9], who instigated the naturalistic approach to mathematics and even suggested that logic could be empirical. However, unlike them, I am inclined towards a more pragmatic theory of truth so, for me, abstract mathematical objects can be called real insofar as they are useful for our scientific reasoning. This evades the problem of trying to find direct referrents of mathematical objects in the physical world. Instead, I simply have to explain why they are useful. In order to do this, I shall have to investigate where mathematics comes from. In this vein, I shall argue that human knowledge has the structure of a *scale-free network* and offer a theory of mathematical theory-building that emphasizes reasoning by analogy within this network. This explains how mathematics can become increasingly abstract, whilst maintaining its tether to empirical reality. It also explains why abstract areas of mathematics that are developed in seeming isolation from physics often show up later in our physical theories.

My title, "Mathematics is Physics", is deliberately chosen in contrast to our dear leader Max Tegmark's Mathematical Universe Hypothesis [10] (see Fig. 1 for a comparison of the two views). This asserts that our universe is nothing but a mathematical structure and that all possible mathematical structures exist in the same sense as our universe. The first part of the hypothesis may be condensed to "Physics is Mathematics", so in this sense I am arguing for precisely the opposite. I will compare and contrast the two ideas at the end.

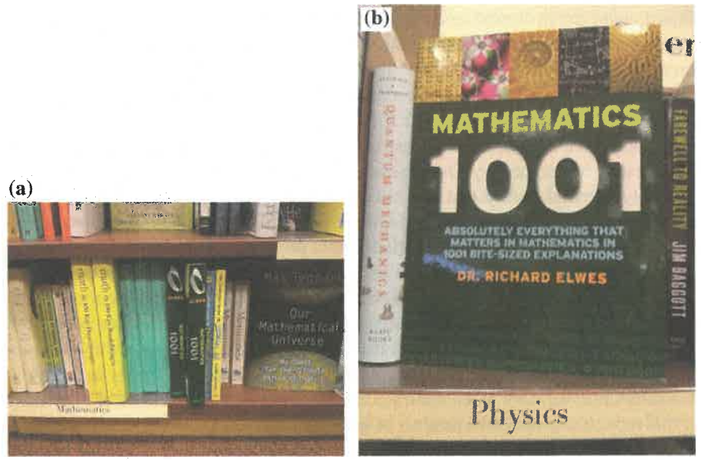

Fig. 1 **a** The staff at my local bookstore agree with Tegmark's Mathematical Universe Hypothesis, as they have displayed his book about the nature of our physical universe in the mathematics section. **b** In response, I placed a mathematics book in the physics section, in order to illustrate my counterhypothesis

What Is Mathematics?

The philosophy of mathematics is now a vast and sprawling subject,[1] so I cannot possibly cover all views of the nature of mathematics here. However, it is useful to introduce two of the more traditional schools of thought—mathematical platonism and formalism—to serve as foils to the naturalistic theory I then develop.

Mathematical Platonism

Mathematical platonism is the idea that mathematical objects are abstract entities that exist objectively and independently of our minds and the physical universe. For example, the geometric concept of a straight line does not refer to any approximation of a straight line that we might draw with pencil on paper. If we view any real world approximation through a microscope we will see that it has rough edges and a finite thickness. On the other hand, a geometric straight line is perfectly straight and has no thickness. It is a fundamentally one-dimensional object, which we can only ever approximate in our three-dimensional universe. Mathematical objects like straight

[1] See [11] for an overview of the subject.

lines do not exist in our universe, so the platonist asserts that they exist in an abstract realm, somewhat akin to the Platonic world of forms. They further assert that we have direct access to this realm via our mathematical intuition.

Mathematical platonism is in direct conflict with naturalism. A naturalistic theory has no place for a dualistic mind that is independent of the structure of our brains. Therefore, if we have intuitive access to an abstract realm, our physical brains must interact with it in some way. Our best scientific theories contain no such interaction.[2] The only external reality that our brains interact with is *physical* reality, via our sensory experience. Therefore, unless the platonist can give us an account of where the abstract mathematical realm actually is in physical reality, and how our brains interact with it, platonism falls afoul of naturalism.

Furthermore, our brains are the product of evolution by natural selection, so, on the naturalist view, whatever mathematical intuitions we have are either a reflection of what was useful for survival, or products of the general intelligence that evolution has endowed us with. Our evolutionary instincts are often poor guides to reality, so it is difficult to see how mathematics could be objective if mathematical intuition is of this type. If it is instead a product of general intelligence then mathematics could either be akin to a creative work of fiction, or it must be the result of reasoning about the physical world that we find ourselves in. Only the latter can explain why our theories of physics are mathematical, so this account should be preferred.

Formalism

Formalism is the view that mathematics is just a game about the formal manipulation of symbols. We specify some symbols, such as marks on a sheet of paper, and rules for deriving one string of symbols from another. Anything that can be derived from those rules is a theorem of the resulting mathematical system.

As a toy example, we could posit that the symbols are 0 and 1. The rules are that you may replace the empty string with either 0 or 1, you may replace any string s ending in 0 with $s1$, and any string s ending in 1 with $s0$. $10 \rightarrow 1010101$ is an example of a (very boring) theorem in this (very boring) formal system.

Formalism has the advantage that it untethers mathematics from an abstract objective realm that is independent of our minds. It is therefore naturalistic in the sense that it does not posit a supernatural world. Mathematics instead becomes an intellectual game, in which we may posit any rules we like. However, formalism has two difficult obstacles to deal with. Firstly, mathematicians do not study arbitrary formal systems. There is no "adding zeroes and ones to the end of binary strings" research group in any mathematics department. Formalists must specify which formal systems count as mathematics. Why is group theory a branch of mathematics, but adding zeroes and ones to the end of binary strings is not? Secondly, even if this is achieved, the formalist has no explanation of why abstract branches of mathematics show up in physics.

[2] This argument is known as Benacerraf's epistemological objection to mathematical platonism [12].

Mathematics as a Natural Science

In response to the objections to platonism and formalism, I wish to defend the idea that mathematics is a natural science, i.e. its subject matter is ultimately our physical universe. To do this, I will borrow one idea each from platonism and formalism. Along with the platonists, I want to view mathematics as being about an objective world that exists independently of us. The difference is that, in my case, this is just the actual physical world that we live in, rather than some hypothetical abstract world of forms.

However, mathematical objects are more abstract than those that appear in the physical world, and they include entities that seem to have no physical referrent, such as hierarchies of infinities of ever increasing size. To deal with this issue, I maintain that mathematical objects do not refer directly to things that exist in the physical universe. As the formalists suggest, mathematical theories are just abstract formal systems, but not all formal systems are mathematics. Instead, mathematical theories are those formal systems that maintain a tether to empirical reality through a process of abstraction and generalization from more empirically grounded theories, aimed at achieving a pragmatically useful representation of regularities that exist in nature.

It is relatively easy to defend the idea that elementary concepts like the finite natural numbers are theories of things that actually exist in the world, viz. the rules of arithmetic are derived from what actually happens if you combine collections of discrete objects such as sheep, rocks, or apples. However, the advanced theories of mathematics deal with entities that are not related to the physical world in any obvious way at all. For example, nobody has ever seen an infinite collection of sheep of any type, let alone one that has the cardinality of some specific transfinite number. Moreover, mathematical theories seem to have their own autonomy, independent of the natural sciences. Pure mathematicians have their own programs of research, with well-motivated questions that are internal to their mathematical theories, making no obvious reference to empirical reality. In response to this, some naturalists are content to only deal with the mathematics that actually occurs in science, and it has been suggested that the theory of functions of real variables is sufficient for this. However, much more abstract mathematics than this appears in modern physics and, in any case, if we are to explain why the fruits of pure mathematics research so often appear in physics at a later date, then we are going to need a theory that encompasses purely abstract research. So, for me, the biggest problem is to explain how abstract pure mathematical theories can be empirical theories in disguise, and to do so in such a way that the later application of these theories to physics becomes natural.

This question cannot be answered without looking at where mathematical theories come from. If I can argue that mathematical theories maintain an appropriate tether to empirical reality then it should be no surprise that the regularities encoded within them, which already refer to nature, later show up in natural science. Developing a theory of human knowledge and mathematical theory-building that does this is the main challenge for my approach.

The Structure of Human Knowledge

Although we are concerned with how mathematics relates to the physical world, it is important to realize that all our knowledge is *human* knowledge, i.e. it is discovered, organized, learned, and evaluated in an ongoing process by a social network of finite beings. This means that the structure of our knowledge will, in addition to reflecting the physical world, also reflect the nature of the process that generates it. By this I do not mean to imply that human knowledge is merely a social construct—it is still knowledge about an objective physical world. However, if we are to explain why abstract pure mathematics later shows up in physics, we are going to have to examine the motivations and methodology of those who develop that mathematics. We have to uncover a hidden empirical tether in their methods and explain how it can be that the patterns and regularities they are studying are in fact patterns and regularities of nature in disguise.

It is common to view the structure of human knowledge as hierarchical. The various attempts to reduce all of mathematics to logic or arithmetic reflect a desire view mathematical knowledge as hanging hierarchically from a common foundation. However, the fact that mathematics now has multiple competing foundations, in terms of logic, set theory or category theory, indicates that something is wrong with this view.

Instead of a hierarchy, we are going to attempt to characterize the structure of human knowledge in terms of a network consisting of nodes with links between them (see Fig. 2). Roughly speaking, the nodes are supposed to represent different fields of study. This could be done at various levels of detail. For example, we could draw a network wherein nodes represent things like "physics" and "mathematics", or we could add more specific nodes representing things like "quantum computing" and "algebraic topology". We could even go down to the level having nodes representing individual concepts, ideas, and equations. I do not want to be too precise about where to set the threshold for a least digestible unit of knowledge, but to avoid triviality it should be set closer to the level of individual concepts than vast fields of study.

Next, a link should be drawn between two nodes if there is a strong connection between the things they represent. Again, I do not want to be too precise about what this connection should be, but examples would include an idea being part of a wider theory, that one thing can be derived from the other, or that there exists a strong direct analogy between the two nodes. Essentially, if it has occurred to a human being that the two things are strongly related, e.g. if it has been thought interesting enough to

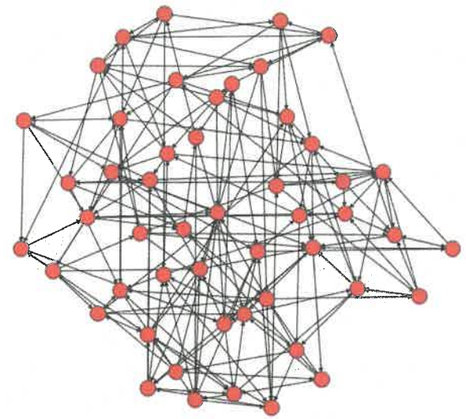

Fig. 2 An example of a network with nodes (*red circles*) and links between them (*arrows*). This network was generated by the SocNetV software package [13]

do something like publish an academic paper on the connection, and the connection has not yet been explained in terms of some intermediary theory, then there should be a link between the corresponding nodes in the network.

If we imagine drawing this network for all of human knowledge then it is plausible that it would have the structure of a *scale-free network* [14]. Without going into technical details, scale-free networks have a small number of *hubs*, which are nodes that are linked to a much larger number of nodes than the average. This is a bit like the 1 % of billionaires who are much richer than the rest of the human population. If the knowledge network is scale-free then this would explain why it seems so plausible that knowledge is hierarchical. In a university degree one typically learns a great deal about one of the hubs, e.g. the hub representing fundamental physics, and a little about some of the more specialized subjects that hang from it. As we get ever more specialized, we typically move away from our starting hub towards more obscure nodes, which are nonetheless still much closer to the starting hub than to any other hub. The local part of the network that we know about looks much like a hierarchy, and so it is not surprising that physicists end up thinking that everything boils down to physics whereas sociologists end up thinking that everything is a social construct. In reality, neither of these views is right because the global structure of the network is not a hierarchy.

As a naturalist, I should provide empirical evidence that human knowledge is indeed structured as a scale-free network. The best evidence that I can offer is that the structure of pages and links on the World Wide Web and the network of citations to academic papers are both scale free [14]. These are, at best, approximations of the true knowledge network. The web includes facts about the Kardashian family that I do not want to categorize as knowledge, and not all links on a website indicate a strong connection, e.g. advertising links. Similarly, there are many reasons why people cite

papers other than a direct dependence. However, I think that these examples provide evidence that the information structures generated by a social network of finite beings are typically scale-free networks, and the knowledge network is an example of such a structure.

A Theory of Theory-Building

We are now at the stage where I can explain where I think mathematical theories come from. The main idea is that when a sufficiently large number of strong analogies are discovered between existing nodes in the knowledge network, it makes sense to develop a formal theory of their common structure, and replace the direct connections with a new hub, which encodes the same knowledge more efficiently.

As a first example, consider the following just-so story about where natural numbers and arithmetic might have come from. Initially, people noticed that discrete quantities of sheep, rocks, apples, etc. all have a lot of properties in common. Absent a theory of this common structure, the network of knowledge has a vast number of direct connections between the corresponding nodes (see Fig. 3). It therefore makes sense to introduce a more abstract theory that captures the common features of all these things, and this is where the theory of number comes in. A vast array of individual connections is replaced by a new hub, which has the effect of organizing knowledge more efficiently. Now, instead of having to learn about quantities of sheep, rocks, apples, etc. individually and then painstakingly investigate each analogy, one need only learn about the theory of number and then apply it to each individual case as needed. In this way, the theory of number remains essentially empirical. It is about regularities that exist in nature, but is removed from the empirical data by one layer of abstraction compared to our direct observations.

Once it is established, the theory of number allows for the introduction of new concepts that are not present in finite collections of sheep, such as infinite sequences and limits. As the theory is more abstract than the empirical phenomena it is derived from, it develops its own internal life and is partially freed from its empirical ties. Such internal questions are sometimes settled by pragmatic considerations, e.g. which definitions make for the most usable extension of the theory beyond strictly finite quantities, but mostly by the way in which the theory ends up interacting with the larger structure of mathematical knowledge. For example, the question of how to define limits of infinite sequences was not really settled until those definitions were needed to understand calculus and analysis, which are themselves abstractions of the physically rooted geometries and flows that exist in our universe.

Once several abstract theories have been developed, the process can continue at a higher level. For example, category theory was born out of the strong analogies that exist between the structure preserving maps in group theory, algebraic topology, and homological algebra [15]. At first sight, it seems like this is a development that is completely internal to pure mathematics, but really what is going on is that mathematicians are noticing regularities, within regularities, within regularities…, within

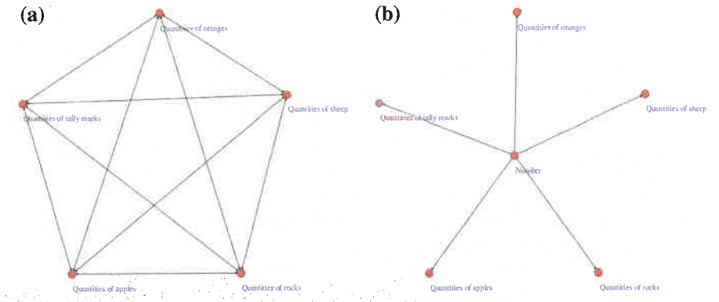

Fig. 3 How abstract mathematical theories arise from analogies between theories at a lower level of abstraction, which are closer to the empirical data. **a** Absent a theory of number, there are a large number of direct analogies between quantities of discrete objects like sheep, rocks, apples, etc. Only analogies between five types of discrete objects are depicted here. There would be vastly more in the real knowledge network. **b** A new hub is introduced to capture the common structure indicated by the analogies. The theory of number captures the regularity at one higher level of abstraction

regularities of the physical world. In this way, mathematics can become increasingly abstract, and develop its own independent structure, whilst maintaining a tether to the empirical world.

Now, to be sure, in the course of this process mathematicians have to make several pragmatic choices that seem to be independent of physics. For example, is it better to reject the axiom of choice, which says that when you have several sets of objects (including a possibly infinite number of sets) then there is a way of picking one object from each of them, or is it better to accept the counter-intuitive implication of this that if you peel a mathematically idealized orange then the peel can be used to completely cover two mathematically idealized oranges of the same size?[3] Ultimately, the axiom of choice is accepted by most mathematicians because it leads to the most useful theories. The theories that employ it lead to a more efficient knowledge network than those that reject it, and this trumps any apparent physical implausibility. In my view, intuition for efficient knowledge structure, rather than intuition about an abstract mathematical world, is what mathematical intuition is about.

It is now straightforward to explain why abstract mathematical theories show up so often in physics. Abstract mathematical theories are about regularities, within regularities, within…, within regularities of our physical world. Physical theories are about exactly the same thing. The only difference is that whilst mathematics started from empirical facts that only required informal observations, physics includes the much more accurate empirical investigations that only became possible due to scientific and technological advances, e.g. the development of telescopes and particle colliders. Nonetheless, it should be no surprise that regularities that are about the selfsame physical world turn out to be related. It should also be no surprise that

[3] This is the Banach–Tarski paradox [16].

pure mathematicians often develop the mathematics relevant for physics a long time before it is needed by physicists, as they have had much more time to investigate regularities at a higher level of abstraction.

Implications

Before concluding, I want to describe two implications of my theory of knowledge and mathematics for the future of physics.

Firstly, in network language, the concept of a "theory of everything" corresponds to a network with one enormous hub, from which all other human knowledge hangs via links that mean "can be derived from". This represents a hierarchical view of knowledge, which seems unlikely to be true if the structure of human knowledge is generated by a social process. It is not impossible for a scale-free network to have a hierarchical structure like a branching tree, but it seems unlikely that the process of knowledge growth would lead uniquely to such a structure. It seems more likely that we will always have several competing large hubs and that some aspects of human experience, such as consciousness and why we experience a unique present moment of time, will be forever outside the scope of physics.

Nonetheless, my theory suggests that the project of finding higher level connections that encompass more of human knowledge is still a fruitful one. It prevents our network from having an unwieldy number of direct links, allows us to share more common vocabulary between fields, and allows an individual to understand more of the world with fewer theories. Thus, the search for a theory of everything is not fruitless; I just do not expect it to ever terminate.

Secondly, my theory predicts that the mathematical representation of fundamental physical theories will continue to become increasingly abstract. The more phenomena we try to encompass in our fundamental theories, the further the resulting hubs will be from the nodes representing our direct sensory experience. Thus, we should not expect future theories of physics to become less mathematical, as they are generated by the same process of generalization and abstraction as mathematics itself.

Conclusions

I have argued that viewing mathematics as a natural science is the only reasonable way of understanding why mathematics plays such a central role in physics. I have also offered a theory of mathematical theory-building that can explain how mathematical theories maintain a tether to the physical world, despite becoming ever more abstract.

To conclude, I want to contrast my theory with Tegmark's Mathematical Universe Hypothesis. In my theory, abstract mathematics connects to the physical world via our direct empirical observations. The latter are at the very edges of the knowledge network, as far away from our most abstract mathematical theories as they

possibly could be. They are the raw material from which our mathematical theories are constructed, but the theories themselves are just convenient representations of the regularities, within regularities, within..., within regularities of the physical world. Thus, my view is opposite to Tegmark's. Mathematics is constructed out of the physical world rather than the other way round.

Like my proposal, the Mathematical Universe Hypothesis is naturalistic. It asserts that there is no abstract mathematical realm independent of our physical universe, because the physical world is *identified* with the realm of all possible mathematical structures. However, the universe we find ourselves in is just one structure in a multiverse of equally real possibilities, and Tegmark's theory does not explain how we could come to know about these other mathematical structures. Indeed, the universes within a multiverse should not have a strong interaction with each other, otherwise they would not be identifiable as independent universes, so it is difficult to see how there could be any causal connection to explain our abstract mathematical theorizing.

The main evidence for the Mathematical Universe Hypothesis is that physics is becoming ever more abstract and mathematical, so it looks like the world described by physics can be identified with a mathematical structure. However, my theory provides an explanation for the increasing abstraction of physics, and can also account for mathematical theory-building in a natural way so, at present, I think it should be preferred.

Addendum

I have been asked to update this essay in light of the discussion that occurred on the FQXi website during the competition. I cannot possibly cover all of the comments, so I shall focus on a few that I consider interesting and accessible. For more details, it is worth reading the comment thread in its entirety [2].

Since Sylvia Wenmackers' winning essay also argues for a naturalistic approach to mathematics [1], I also thought it worthwhile to discuss how my position differs from hers. We start with that before moving on to the website comments.

Response to Sylvia Wenmackers' Essay

The main advantage of a naturalistic view of mathematics is that it offers a simple explanation of the alleged unreasonable effectiveness of mathematics in physics. If mathematics is fundamentally about the physical world then it is no surprise that it occurs in our description of the physical world.

However, the naturalistic view is prima facie absurd because mathematical truth seems to have nothing to do with physical world. The main task of a naturalistic account of mathematics is therefore a deflationary one: explain why mathematical theories are empirical theories in disguise and why we have been so easily misled

into thinking this is not the case. Wenmackers gives a battery of arguments for this position based on four "elements". Here, I will focus on the first and third elements, which are based on evolution by natural selection and selection bias respectively, because this is where I disagree with her. I more or less agree with her discussion of the other two elements.

The Evolutionary Argument

The evolutionary argument asserts that our cognitive abilities, and in particular our ability to do mathematics, are the result of evolution by natural selection. It is therefore no surprise that they reflect physical reality, as physical reality provided the environmental pressures that selected for those abilities.

This argument works well for basic mathematics, such as arithmetic and elementary pattern recognition. The ability to distinguish a tree that has five apples growing on it from one that has two has an obvious evolutionary advantage.

However, our main task is to explain why our most advanced and abstract theories of mathematics crop up so often in modern physics, not just the basic theories like arithmetic, and there is no conceivable evolutionary pressure towards understanding the cosmos on a large scale. Knowing the ultimate fate of the universe may well be crucial for our (very) long term survival, but cosmology operates on a much longer timescale than evolution by natural selection, so, for example, there is no immediate environmental pressure towards discovering general relativity, nor the differential geometry needed to formulate it.

On the other hand, it happens that evolution has endowed us with general curiosity and intelligence. This does have a survival advantage as, for example, a species that is able to detect patterns in predator attacks and pass that knowledge on to the next generation without waiting for genetic changes to make the knowledge innate can adapt to its environment more quickly. Such curiosity and intelligence are not the inevitable outcome of evolution, as the previous dominance of dinosaurs on this planet demonstrates, but just one possible adaptation that happened to occur in our case. Like many adaptations, it has side effects that are not immediately related to our survival, one of which is that some of us like to think about the large scale cosmos.

Since our general curiosity and intelligence are only a side effect of adaptation, so are modern physics and mathematics. Therefore, I find it puzzling to argue for the efficacy of our reasoning in these areas based on evolution. Evolution often endows beliefs and behaviours that are good heuristics for the cases commonly encountered, but a poor reflection of reality (consider optical illusions for example). With general intelligence one can, with considerable effort, reason oneself out of such beliefs and behaviours. This is why I stated that mathematical intuition must be a product of general intelligence rather than a direct evolutionary adaptation in my essay.

I think that understanding how a network of intelligent beings go about organizing their knowledge is at the root of the efficacy of mathematics in physics. It should not matter whether those beings are the products of evolution by natural selection or some hypothetical artificial intelligences that we may develop in the future. For this reason,

I take the existence of intelligent beings as a starting point, rather than worrying about how they got that intelligence.

Selection Bias

Wenmackers' selection bias argument is an attempt to deflate the idea that mathematics is unreasonably effective in physics. The idea is that, due to selection bias, we tend to remember and focus on those cases in which mathematics was successfully applied in physics, whereas the vast majority of mathematics is actually completely useless for science. Here is the argument in her own words.

> Among the books in mathematical libraries, many are filled with theories for which not a single real world application has been found. We could measure the efficiency of mathematics for the natural science as follows: divide the number of pages that contain scientifically applicable results by the total number of pages produced in pure mathematics. My conjecture is that, on this definition, the efficiency is very low. In the previous section we saw that research, even in pure mathematics, is biased towards the themes of the natural sciences. If we take this into account, the effectiveness of mathematics in the natural sciences does come out as unreasonable - unreasonably low, that is.

My first response to this is to question whether the efficiency is actually all that low. After all, the vast majority of pages written by theoretical physicists might also be irrelevant to reality, and these are people who are deliberately trying to model reality. We need only consider the corpus of mechanical models of the ether from the 1800s to render this plausible, let alone the vast array of current speculative models of cosmology, particle physics, and quantum gravity. It is not obvious to me whether the proportion of applicable published mathematics is so much lower than the proportion of correct published physics, and, if it is not, then a raw page count does not say much about the applicability of mathematics in particular.

Even if the efficiency of mathematics is much lower than that of physics, it not obvious how low an inefficiency would be unreasonably low. If mathematics were produced by monkeys randomly hitting the keys of typewriters then the probability of coming up with applicable mathematics would be ridiculously small, akin to a Boltzmann brain popping into existence via a fluctuation from thermal equilibrium. In light of such a ridiculously tiny probability, an efficiency of say 0.01 %, which looks small from an everyday point of view, would indicate an extremely high degree of unreasonable effectiveness. Of course, mathematicians are not typewriting monkeys, but unless one is already convinced that the development of mathematics is correlated with the development physics by one of Wenmackers' other arguments, then even a relatively tiny efficiency could seem extremely improbable.

My second response to the selection bias argument is that mathematics is not identical to the corpus of mathematical literature laid out in a row. Some mathematical theories are considered more important than others, e.g. the core topics taught in an undergraduate mathematics degree. Therefore, mathematical theories ought to be weighted with their perceived importance when calculating the efficiency of mathematics. If you buy my network model of knowledge then the number of inbound

links to a node could be used to weight its importance, as in the Google Page rank algorithm. I would conjecture that the efficiency of mathematics is much higher when weighted by perceived importance. I admit that this argument could be accused of circularity because one of the reasons why an area of mathematics might be regarded as important is its degree of applicability. However, this just reinforces the point that mathematics not an isolated subject, but must be considered in the context of the whole network of human knowledge.

Responses to Comments on the FQXi Website

Other Processes in the Knowledge Network

Several commenters expressed doubts that my theory of theory building captures everything that is going on in mathematics. For example, Wenmackers commented:

> Is this is correct summary of your main thesis (in Section 4)? : "First, humans studied many aspects the world, gathering knowledge. At some point, it made sense to start studying the structure of that knowledge. (And further iterations.) This is called mathematics."
>
> Although I find this idea appealing (and I share your general preference for a naturalistic approach), it is not obvious to me that this captures all (or even the majority) of mathematical theories. In mathematics, we can take anything as a source of inspiration (including the world, our the structure of our knowledge there of), but we are not restricted to studying it in that form: for instance, we may deliberately negate one of the properties in the structure that was the original inspiration, simply because we have a hunch that doing so may lead to interesting mathematics. Or do you see this differently?

There are other processes going on in the knowledge network beyond the theory-building process that I described in my essay. I did not intend to suggest otherwise. The reason why I focused on the process of replacing direct links by more abstract theories is because I think it can explain how mathematics becomes increasingly abstract, whilst maintaining its applicability. But this is clearly not the only thing that mathematicians do.

One additional process that is going on is a certain amount of free play and exploration, as also noted by Ken Wharton in his essay [17]. Mathematical axioms may be modified or negated to see whether they lead to interesting theories. However, as I argued earlier, mathematical theories should be weighted with their perceived importance when considering their place in the corpus of human knowledge. Modified theories that are highly connected to existing theories, or to applications outside of mathematics, will ultimately be regarded as more important. It is possible that a group of pure mathematicians will end up working on a relative backwater for an academic generation or two, but this is likely to be given up if no interesting connections are forthcoming.

For my theory, it is important that these additional processes should not have a significant impact on the broad structure of the knowledge network. There should not be a process where large swaths of pure mathematicians are led to work on

completely isolated areas of the network, developing a large number of internal links that raise the perceived importance of their subject, with almost no links back to the established corpus of knowledge. Personally, I think that any such process is likely to be dominated by processes that do link back strongly to existing knowledge, but this is an empirical question about how the mathematical knowledge network grows. To address it, I would need to develop concrete models, and compare them to the actual growth of mathematics.

What Physical Fact Makes a Mathematical Fact True?

Perhaps the highlight of the comment thread was a discussion with Tim Maudlin. It started with the following question:

> I'm not sure I understand the sense in which mathematics is supposed to be "about the physical world" as you understand it. In one sense, the truth value of any claim about the physical world depends on how the physical world is, that is, it is physically contingent. Had the physical world been different, the truth value of the claim would be different. Now take a claim about the integers, such as Goldbach's conjecture. Do you mean to say that the truth or falsity of Goldbach's conjecture depends on the physical world: if the physical world is one way then it is true and if it is another way it is false? What feature of the physical world could the truth or falsity of the conjecture possibly depend on?

I stated in the essay that I think mathematical theories are formal systems, but not all formal systems are mathematics. The role of physics is to help delineate which formal systems count as mathematics. Therefore, if Goldbach's conjecture is a theorem of Peano arithmetic then I would say it is true in Peano arithmetic. If we want to ask whether it is true of the world then we have to ask if Peano arithmetic is the most useful version of arithmetic by looking how it fits into the knowledge network as a whole. It may be that more than one theory of arithmetic is equally useful, in which case we may say that it has indefinite truth value, or we may want to say that it is true in one context and not another if the two theories of arithmetic have fairly disjoint domains of applicability.

If the Goldbach conjecture is not provable in Peano arithmetic, but is provable in some meta-theory, then we can ask the same questions at the level of the meta-theory, i.e. is it more useful than a different meta-theory. This sort of consideration has happened in mathematics in a few cases, e.g. most mathematicians choose to work under the assumption that the axiom of choice is true, mainly, I would argue, because it leads to more useful theories.

At this point in the discussion, Maudlin accused me of being a mathematical platonist. His point is that if I accept theoremhood as my criterion of truth then I am admitting some mathematical intuitions as self-evident truths, so if my goal is to remove intuition as the arbiter of mathematical truth then I have not yet succeeded. Specifically, in order to even state what it means for something to be a theorem in a formal system, I need to accept at least some of the structure informal logic, e.g. things like: if A is true and B is true then A AND B is also true. Why do we accept

these ideas of informal logic? Primarily because they seem to be self-evidently true, but this is an appeal to unfettered mathematical intuition.

Maudlin points out that, because of this, it is difficult to avoid some form of mathematical platonism, if only about the basic ideas of logic. I am not yet prepared to accept this and would argue, along with Quine and Putnam [6, 8], that logic may be empirical (in my case, I would say that even very basic informal logic may be empirical).

I do not deny that I accept informal logic because it seems self-evident to me. However, as I argued in the essay, my intuitions come from my brain, and my brain is a physical system. Therefore, if I have strong intuitions, they must have been put there by the natural processes that led to the development of my brain. The likely culprit in this case is evolution by natural selection. Organisms that intuitively accept the laws of informal logic survive better than those that do not because those laws are true of our physical universe, so that creates a selection pressure to have them built in as intuitions.

Does this mean there are conceivable universes in which the laws of basic logic are different,[4] e.g. in which Lewis Carroll's modus ponens denying tortoise [18] is correct. I have to admit that I have difficulty imagining it, but it does not seem totally inconceivable. In such a universe, what counts as a mathematical truth would be different. In other words, mathematical truth might be empirical because the laws of logic are.

In addition to this, there is a much more prosaic way in which mathematical truth is dependent on the physical laws. Imagine a universe in which planets do not make circular motions as time progresses, but instead travel in straight lines through a continually changing landscape. Inhabitants of such a planet would probably not measure time using a cyclical system of minutes, hours, days, etc. as we do, but instead just use a system of monotonically increasing numbers. A bit more fancifully, suppose that in this hypothetical universe, every time a collection of twelve discrete objects like sheep, rocks, apples, etc. are brought together in one place they magically disappear into nothingness. Inhabitants of this world would use clock arithmetic, technically known as mod 12 arithmetic, to describe collections of discrete objects. Their view of how to measure time versus how to measure collections of discrete objects would be precisely the reverse of ours. At least one of the senses in which $12 + 1 = 13$ in our universe is not true in theirs, and I would say that this is a sense in which mathematical truth depends on the laws of physics.

It is fair to say that Maudlin was not impressed by this example, but I take it deadly seriously. What counts as mathematics and what counts as mathematical truth are, in my view, pragmatically dependent on how our mathematical theories fit into the structure of human knowledge. If the empirical facts change then so does the structure of this network. The meaning of numbers, in particular, is dependent on how collections of discrete objects behave in our universe, and if you change that

[4]The usual terminology for hypothetical universes with different laws is "logically possible". That seems inappropriate here, but the terminology does show how ingrained into our minds the basic laws of logic are.

then you change what makes a given theory of number useful, and hence true in the pragmatic sense. It is this that makes the theory of number a hub in the network of human knowledge and this is what philosophers ought to be studying if they want to understand the meaning of mathematics. The usual considerations in the foundations of mathematics, such as deriving arithmetic from set theory, though still well-connected to other areas of mathematics, are comparative backwaters. If we really want to understand what mathematics is about, we ought to get our heads out of the formal logic textbooks and look out at the physical world.

Why are There Regularities at All?

Sophia Magnusdottir points out that my approach does not address why there are regularities in nature to begin with.

> In a nutshell what you seem to be saying is that one can try to understand knowledge discovery with a mathematical model as well. I agree that one can do this, though we can debate whether the one you propose is correct. But that doesn't explain why many of the observations that we have lend themselves to mathematical description. Why do we find ourselves in a universe that does have so many regularities? (And regularities within regularities?) That really is the puzzling aspect of the "efficiency of mathematics in the natural science". I don't see that you address it at all.

There are two relevant kinds of regularities here: the regularities described by our most abstract mathematical theories on the one hand, and the regularities of nature on the other. On the face of it, these two types of regularity have little to do with one another. The fact that the regularities described by our most abstract mathematical theories so often show up in physics is what I take to be the "unreasonable" effectiveness of mathematics in the physical sciences.

What I have tried to do is to argue that these two types of regularity are more closely connected than we normally suppose. They both ultimately describe regularities, within regularities, ..., within the natural world. I have not even tried to address the question of why there are regularities in nature in the first place. Instead, I have taken their existence as my starting point. If we live in a universe with regularities, the process of knowledge growth is such that what we call mathematics will naturally show up in physics. This answers what I take to be the problem of "unreasonable" effectiveness.

Of course, one can try to go further by asking why there are any regularities in the first place. I do not think that anyone has provided a compelling answer to this, and I suspect that it is one of those questions that just leads to an infinite regress of further "why" questions.

For example, the Mathematical Universe Hypothesis may seem, superficially, to explain the existence of regularities. If our universe literally is a mathematical structure, and mathematical structures describe regularities, then there will necessarily be regularities in nature. However, one can then ask why our universe is a mathematical structure, which is just the same question in a different guise.

If we are to take the results of science seriously, the idea that our universe is sufficiently regular to make science reliable has to be assumed. There is no proof of this, despite several centuries of debate on the problem of induction. Although this is an interesting issue, I doubt that the problem can ever be resolved in an uncontroversial way and it seems, to me at least, to be a different and far more difficult problem than the "unreasonable" effectiveness of mathematics in physics. If my ideas are correct then at least there is now only one type of regularity that needs to be explained.

Elegance or Efficiency?

Alexy Burov made the following point.

> Wigner's wonder about the relation of physics and mathematics is not just abut the fact that there are some mathematical forms describing laws of nature. He is fascinated by something more: that these forms are both elegant, while covering a wide range of parameters, and extremely precise. I do not see anything in your paper which relates to that amazing and highly important fact about the relation of physics and mathematics.

I take the key issue here to be that I have not explained why the mathematics used in physics is "elegant". After all, if we had a bunch of different laws covering different parameter ranges then we could always put them together into a single structure by inserting a lot of "if" clauses into our laws of physics. We can also make them arbitrarily precise by adding lots of special cases in this way. Presumably though, the result of this would be judged "inelegant".

To be honest, I have a great deal of trouble understanding what mathematicians and physicists mean by "elegance" (hence the scare quotes). For this reason, I have emphasized that the mathematics in modern physics is "abstract" and "advanced" rather than "elegant".

A more precise definition of elegance is needed to make any progress on this issue. One concrete suggestions is that perhaps elegance refers to the fact that the fundamental laws of physics are few in number so they can be written on a t-shirt. It is tempting to draw the analogy with algorithmic information here, i.e. the length of the shortest computer program that will generate a given output [19]. Perhaps the laws of physics are viewed as elegant because they have low algorithmic information. We get out of them far more than we put in.

So, perhaps what we call "elegance" really means the smallest possible set of laws that encapsulates the largest number of phenomena. If so, then what we need to explain is why the process of scientific discovery would tend to produce laws with low algorithmic information. The idea that scientists are trying to optimize algorithmic information directly is a logician's parody of a complex social process. Instead, we need to determine whether the processes going on in the knowledge network would tend to reduce the algorithmic information content of the largest hubs in the network. In this I am encouraged by the fact that many scale-free networks exhibit the "small

world" phenomenon in which the number of links in a path connecting two randomly chosen nodes is small.[5] If this is true of the knowledge network then it means that the hubs must be powerful enough to derive the empirical phenomena in a relatively small number of steps. The average path length between two nodes might be taken as a measure of the efficiency with which our knowledge is encoded in the network, or, if you prefer, its "elegance".

Now, of course, this may be completely unrelated to what everyone else means by the word "elegance", as applied to mathematics and physics. If so, a more precise definition, or an analysis into more primitive concepts, is needed before we can address the problem. Once we have that, I suspect the problem might not look so intractable.

References

1. S. Wenmackers. Children of the cosmos. FQXi Essay Contest Entry. Available at http://fqxi.org/community/forum/topic/2492, 2015.
2. Discussion thread for "Mathematics is Physics". http://fqxi.org/community/forum/topic/2364, 2015.
3. Eugene P. Wigner. The unreasonable effectiveness of mathematics in the natural sciences. *Communications on Pure and Applied Mathematics*, 13(1):1–14, 1960.
4. W. V. Quine. Epistemology naturalized. In *Ontological Relativity and Other Essays*, chapter 3, pages 69–90. Columbia University Press, 1969.
5. David Papineau. Naturalism. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosopy*. Spring 2009 edition, 2009. URL: http://plato.stanford.edu/archives/spr2009/entries/naturalism/.
6. Willard Van Orman Quine. Two dogmas of empiricism. *The Philosophical Review*, pages 20–43, 1951.
7. Hilary Putnam. Mathematics without foundations. *The Journal of Philosophy*, 64(1):5–22, 1967. URL: http://www.jstor.org/stable/2024603.
8. Hilary Putnam. Is logic empirical? In Robert S. Cohen and Marx W. Wartofsky, editors, *Boston Studies in the Philosophy of Science*, volume 5, pages 216–241. D. Reidel, 1968.
9. Hilary Putnam. *Philosophy of Logic*. George Allen & Unwin Ltd., 1972.
10. Max Tegmark. *Our Mathematical Universe*. Knopf, 2014.
11. Leon Horsten. Philosophy of mathematics. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Spring 2015 edition, 2015. URL: http://plato.stanford.edu/archives/spr2015/entries/philosophy-mathematics/.
12. P. Benacerraf. Mathematical truth. In P. Benacerraf and Hilary Putnam, editors, *Philosophy of Mathematics: Selected Readings*, pages 403–420. Cambridge University Press, 2nd edition, 1983.
13. Dimitris V. Kalamaras. SocNetV. v1.5. URL: http://socnetv.sourceforge.net.
14. Rka Albert and Albert-Lszl Barabsi. Statistical mechanics of complex networks. *Reviews of Modern Physics*, 74(1):47–96, 2002. eprint arXiv:cond-mat/0106096. doi:10.1103/RevModPhys.74.47.
15. Elaine Landry and Jean-Pierre Marquis. Categories in context: Historical, foundational, and philosophical. *Philosophica Mathematica*, 13(1):1–43, 2005. doi:10.1093/philmat/nki005.
16. Stefan Banach and Alfred Tarski. Sur la dcomposition des ensembles de points en parties respectivement congruentes. *Fundamenta Mathematicae*, 6:244–277, 1924.

[5] Technically, it grows like the logarithm of the number of nodes.

17. K. Wharton. Mathematics: Intuition's consistency check. FQXi Essay Context Entry. Available at http://fqxi.org/community/forum/topic/2493, 2015.
18. L. Carroll. What the tortoise said to Achilles. Mind, 104(416):691–693, 1895. doi:10.1093/mind/104.416.691.
19. P. D. Grunwald and P. M. B. Vitanyi. Algorithmic information theory. In P. Adriaans and J. van Benthem, editors, *Philosophy of information*, Handbook of the philosophy of science, chapter 3c, pages 281–317. Elsevier, 2008. arXiv:0809.2754, doi:10.1016/B978-0-444-51726-5.50013-3.

My God, It's Full of Clones: Living in a Mathematical Universe

Marc Séguin

Abstract Imagine there's only math—physics is nothing more than mathematics, we are self-aware mathematical substructures, and our physical universe is nothing more than a mathematical structure "seen from the inside": that's what Max Tegmark's *Mathematical Universe Hypothesis* (MUH) proposes. In this paper, I will discuss some consequences of the MUH. While Tegmark claims that the MUH implies the existence of an enormous but *finite* multiverse (to avoid the measure problem that occurs when you try to calculate probabilities within an infinite ensemble), I will argue that it implies the existence of the largest imaginable multiverse, the *Maxiverse*, where every imaginable conscious observation is guaranteed to happen. I will attempt to explain why, of all the worlds in the Maxiverse, we happen to live in one that can be understood by physical laws simple enough to be discovered (or, at least, approximated well enough for predictive and technological purposes). I will explore the issue of personal identity in the context of a Maxiverse that contains an infinite number of exact clones of myself, and discuss the *Maxiverse Immortality Argument*, the idea that I should expect my future subjective experience to be unbounded. Finally, I will consider the question of whether or not the Maxiverse hypothesis can make predictions that can be put to the test.

Math from Nothing and Your Physics for Free

What is the relationship between mathematics, the study of abstract structures, and physics, the study of the physical world? Since the theories of modern physics are formulated in the language of mathematics, we can try to express the relationship between mathematics and physics by the equation

$$Physics = Math + ?$$

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