Quantum Foundations Lecture 22

April 25, 2018

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HSC112

Announcements

- Assignments: Final Version due May 2.
- Homework 4 due April 30.

Conditional Independence

- Two random variables, A and B are *independent,* denoted $A \perp B$ if
$$P(A, B) = P(A)P(B)$$
- The *conditional probability* of B given A is
$$P(B|A) = \frac{P(A, B)}{P(A)}$$
- Independence can equivalently be written as
$$P(B|A) = P(B) \qquad \text{or} \qquad P(A|B) = P(A)$$
- Two random variables, A and B are *conditionally independent* given C, denoted $A \perp B|C$ if any of the following three equivalent conditions holds
  1. $P(A|B, C) = P(A|C)$
  2. $P(B|A, C) = P(B|C)$
  3. $P(A, B|C) = P(A|C)P(B|C)$

Reichenbach's Principle

- Scientific realists usually think that correlations need to have causes.
- *Reichenbach's principle* encapsulates how this is supposed to work.

  - If A and B are correlated $P(A, B) \neq P(A)P(B)$ then either:

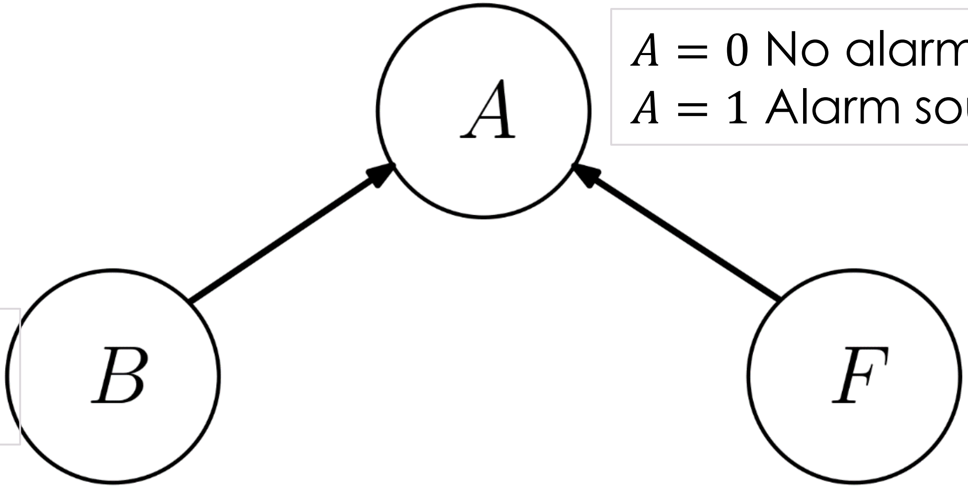
1. A is the cause of B
2. B is the cause of A
3. There is a common cause C for both A and B, and $A \perp B | C$

$$P(A, B|C) = P(A|C)P(B|C)$$

The Markov Condition

- Reichenbach's principle can be formulated in the language of *Causal (Bayesian) Networks*.

$A = 0$ No alarm
$A = 1$ Alarm sounding



$B = 0$ No burglar in house
$B = 1$ Burglar in house

$F = 0$ No fire in house
$F = 1$ Fire in house

$$P(A, B, F) = P(A|B, F)P(B)P(F)$$

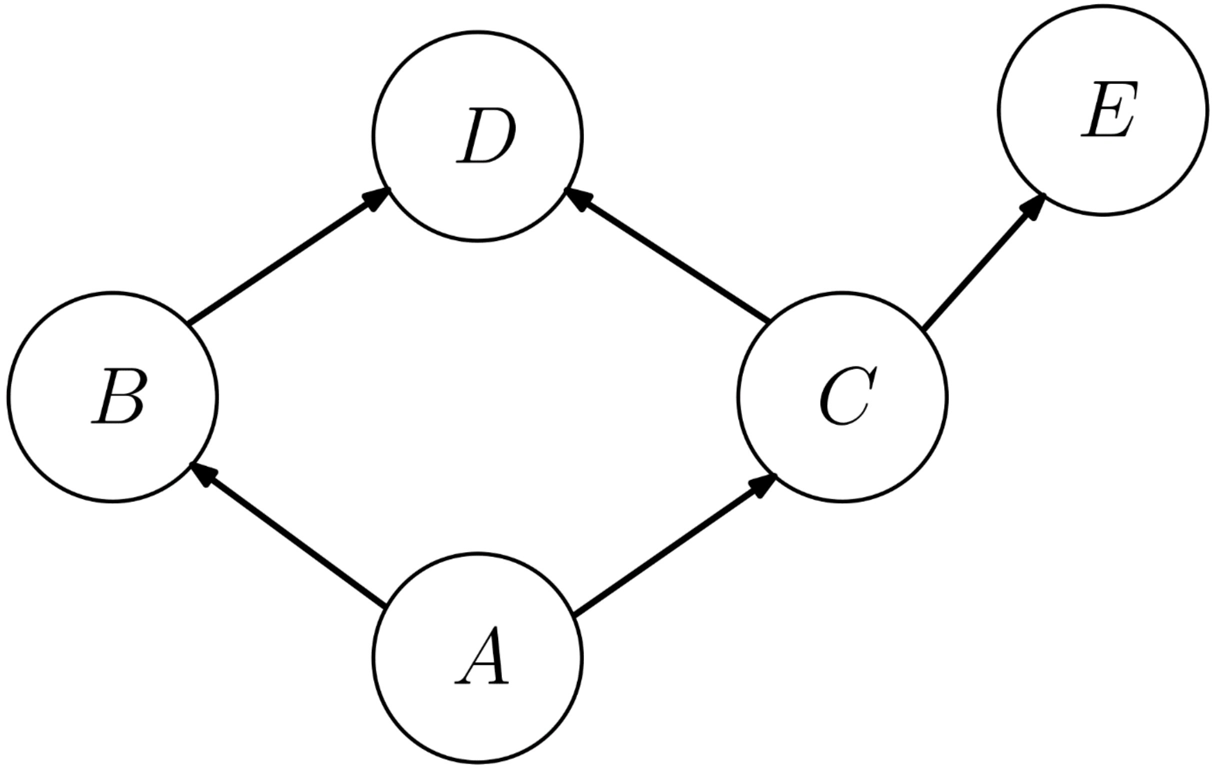
The Markov Condition

- We draw a *directed acyclic graph:*
  - The vertices are the random variables.
  - We draw an edge from A to B if A is a direct cause of B.
  - The probabilities factor according to the *Markov Condition*

$$P(X_1, X_2, \cdots, X_n) = P(X_n|\mathrm{pa}(X_n)) \cdots P(X_2|\mathrm{pa}(X_2))P(X_1|\mathrm{pa}(X_1))$$

where $\mathrm{pa}(X)$ denotes the parents of X in the graph.

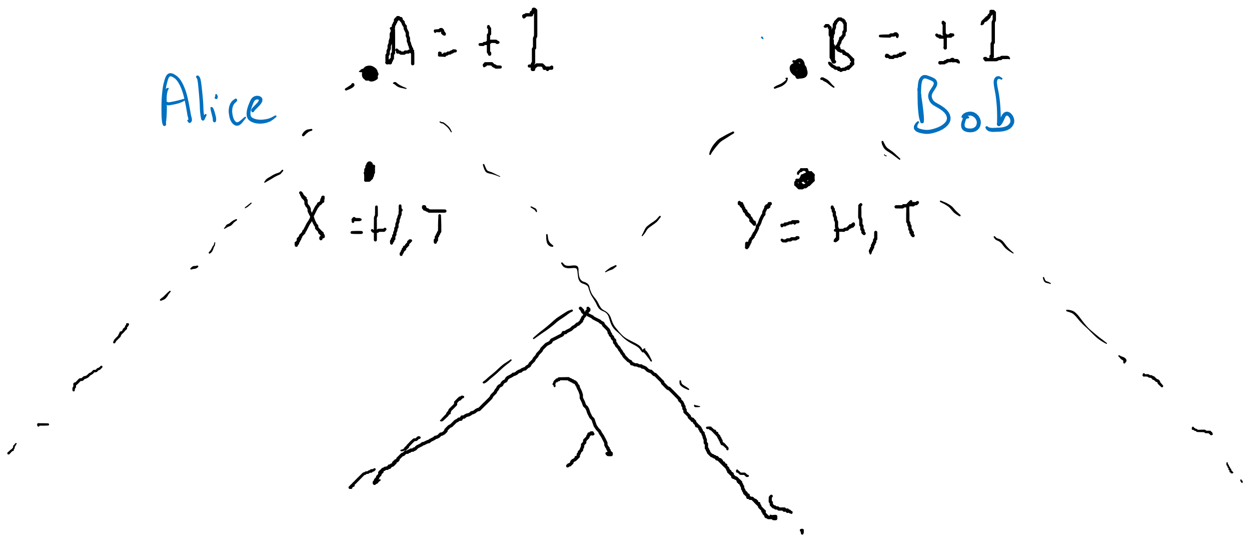
Another Example



$$P(A, B, C, D, E) = P(E|C)P(D|B, C)P(C|A)P(B|A)P(A)$$

Application to Bell Experiments

- Suppose Alice's coin flip and answer happen at spacelike separation to Bob's coin flip and answer.



- Since Alice and Bob's wings of the experiment are spacelike separated, according to special relativity (X, A) cannot be direct causes of (Y, B) and vice versa.

- Let λ be a complete description of the state of affairs in a region that screens off (X, A) from (Y, B)
  - Any lightlike path from (X, A) to (Y, B) via the past must intersect the region.

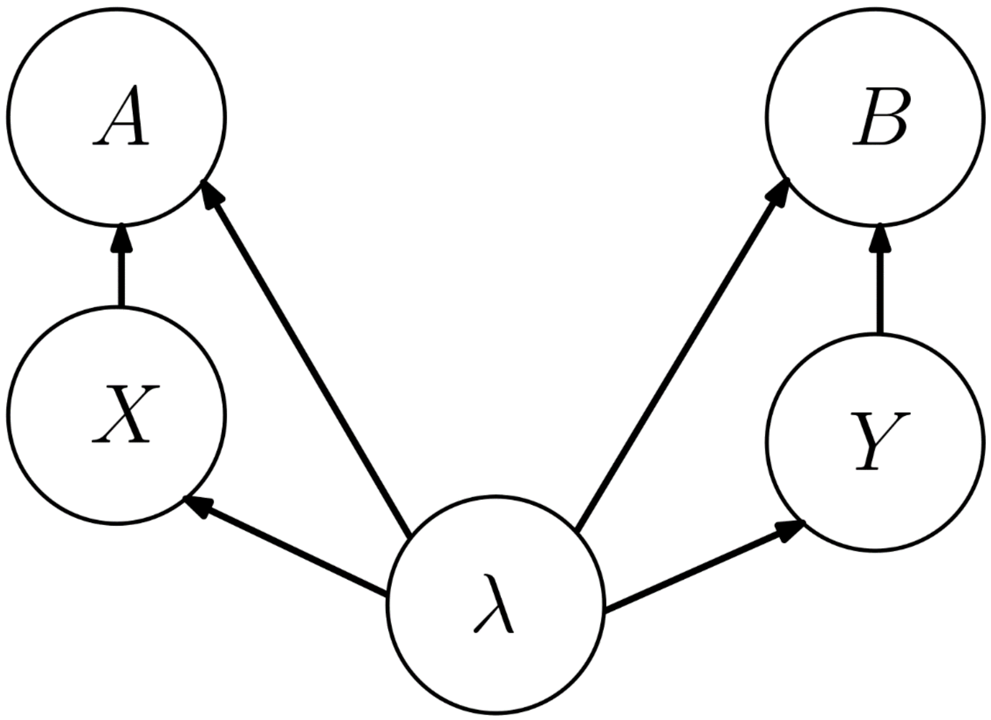
- ⇒ Any common cause of (X, A) and (Y, B) must be contained in λ.

Application to Bell Experiments

- According to special Relativity, the possible causal relationships are:

$$P(A, B, X, Y, \lambda)$$

$$= P(B|Y, \lambda)P(A|X, \lambda)P(Y|\lambda)P(X|\lambda)P(\lambda)$$



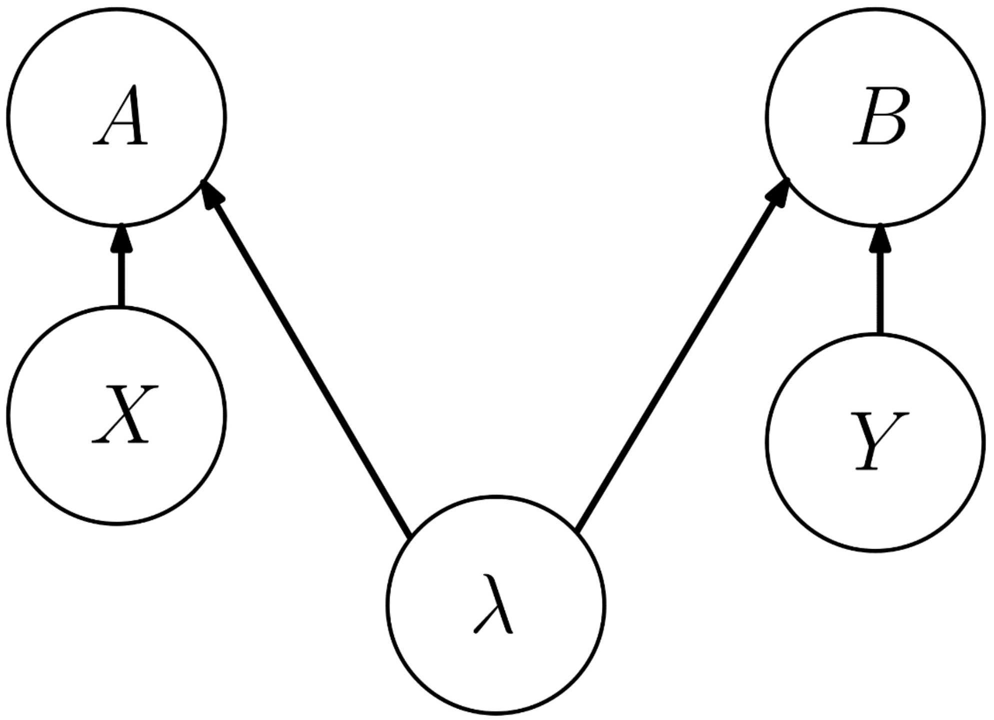
Application to Bell Experiments

- However, we normally assume that the coin flips X and Y are freely chosen, independently from the system being measured.
- This leads to the *measurement independence* assumption

$$X, Y \perp \lambda$$
$$P(X, Y|\lambda) = P(X, Y)$$

- With this, we have

$$P(A, B, X, Y, \lambda)$$
$$= P(B|Y, \lambda)P(A|X, \lambda)P(Y)P(X)P(\lambda)$$



Application to Bell Experiments

$$P(A,B,X,Y,\lambda) = P(B|Y,\lambda)P(A|X,\lambda)P(Y)P(X)P(\lambda)$$

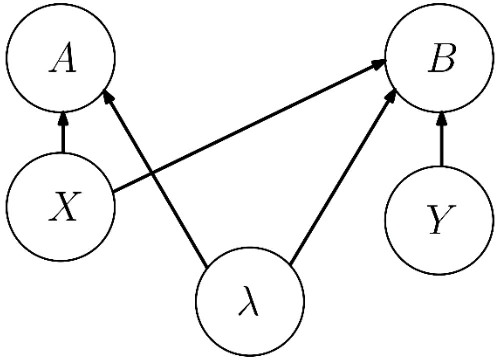
- If we conditionalize on X, Y and λ, we get
$$P(A,B|X,Y,\lambda) = P(B|Y,\lambda)P(A|X,\lambda)$$
- This condition is known as *local causality*
- To reiterate, it follows from:
  - The Markov condition (Reichenbach's principle)
  - The causal structure given by special relativity (spacelike separation)
  - The assumption that X and Y are chosen independently of the system being investigated.

Application to Bell Experiments

- If we now compute the observed conditional probabilities, we will get
$$P(A,B|X,Y) = \sum_{\lambda} P(B|Y,\lambda)P(A|X,\lambda)P(\lambda)$$
- Let's think about what this says in terms of the game we discussed last lecture.
  - Alice and Bob get together to determine a joint strategy – call it λ.
  - Based on λ and X, Alice flips a biased coin to determine A with probability $P(A|X,\lambda)$.
  - Based on λ and Y, Bob flips a biased coin to determine B with probability $P(B|Y,\lambda)$.
- But this is exactly the sort of strategy we showed must satisfy the CHSG inequality.
- The quantum violation therefore rules out a locally causal model.

Implications

- If you accept the Markov condition and measurement independence, then there must be a superluminal causal influence (nonlocality). For example:



- Your model violates relativity at the ontological level.

- We could instead reject the Markov condition:
  - Correlations do not have to have causal explanations.
  - This is appealing to anti-realists.
- We could modify the Markov condition:
  - Causal explanations work differently in quantum theory.
- We could reject measurement independence:
  - There is no free choice.
  - Superdeterminism
  - Retrocausality

Summary of Ontological Models

- If our interpretation of quantum mechanics fits into the ontological models framework then it has to have a number of unappealing features:
  - Excess baggage
  - Contextuality
  - ψ-ontology
  - Nonlocality
- Two options:
  - Bite the bullet and adopt an interpretation that has these features, viewing the no-go theorems as justification for why we have to have these features (de Broglie-Bohm, Spontaneous Collapse theories).
  - Go anti-realist or adopt a more exotic ontology that does not fit into the ontological models framework (Copenhagenish, many-worlds).

10) Interpretations of Quantum Theory

i. Continuous Variable Quantum Theory
ii. De Broglie-Bohm Theory
iii. Spontaneous Collapse Theories
iv. Everett/Many-Worlds
v. Copenhagenish Interpretations

10.i) Continuous Variable Quantum Theory

- De Broglie-Bohm and Spontaneous Collapse privilege the position representation of quantum theory, so we will have to quickly review how this works.
- There is a good reason for this:
  - The world around us looks localized in position, i.e. we do not directly experience a chair that is in a superposition of two locations.
  - If we add something to quantum theory that localizes objects in position space, we will be able to explain this and save the phenomena of ordinary experience.
  - Some classical experiences do not seem to be directly related to position, e.g. the voltage in a circuit or my experience of color.
  - However, the claim is that these can always be explained in terms of position, e.g. the position of a needle on a voltmeter or the positions of electrons in my synapses.

Position

- Recall that observables in quantum theory are Hermitian operators. Their eigenvalues are the possible values that can be obtained in a measurement.
- If we want position to be described in this way then we need a Hermitian operator with a continuum of eigenvalues and eigenvectors:

$$\hat{x} = \int_{-\infty}^{+\infty} \mathrm{d}x \; x |x\rangle\langle x|$$

- Compare this to the discrete case

$$\hat{A} = \sum_j a_j |\phi_j\rangle\langle\phi_j|$$

Position

- In the discrete case, the eigenstates $|\phi_j\rangle$ form a complete orthonormal basis, so we can write any state as

$$|\psi\rangle = \sum_j \alpha_j |\phi_j\rangle$$

- We can recover the coefficients α_j via $\alpha_j = \langle\phi_j|\psi\rangle$.
- We want the position eigenstates to form a complete orthonormal basis, so that we can write a state as

$$|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x)|x\rangle$$

where $\psi(x)$ is a function (called the *wavefunction*) that replaces α_j.

The Dirac δ-Function

- We would like to preserve the formula $\psi(x) = \langle x|\psi\rangle$ which generalizes $\alpha_j = \langle \phi_j|\psi\rangle$.
- In order to do this, we need
$$\psi(x) = \langle x|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x'\psi(x')\langle x|x'\rangle$$
- So we need the inner product $\langle x|x'\rangle$ to be a "function" $\langle x|x'\rangle = \delta(x' - x)$ that behaves like:
$$\int_{-\infty}^{+\infty} \mathrm{d}x'\, \psi(x')\delta(x' - x) = \psi(x)$$
- The generalized function $\delta(x' - x)$ is called the *Dirac δ-function*.

The Dirac δ-function

- To recap, the Dirac δ-function $\delta(x)$ is defined by the property
$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \delta(x) f(x) = f(0)$$
for any function $f(x)$.
- Roughly speaking, it takes the value ∞ at $x = 0$ and is zero elsewhere.
- If we have the defining property, then by change of variables we will have
$$\int_{-\infty}^{+\infty} \mathrm{d}x'\, f(x')\delta(x' - x) = f(x)$$
as required.

Position Eigenstates

- The position eigenstates satisfy
$$\langle x|x'\rangle = \delta(x' - x)$$
- Compare this with the discrete case
$$\langle \phi_j|\phi_k\rangle = \delta_{jk}$$
- The Dirac $\delta(x' - x)$ plays the same role as the Kronecker δ_{jk}, but there is an important difference.
  - $\delta_{jj} = 1$, so our orthonormal basis consists of unit vectors.
  - $\delta(0) = \infty$, so $|x\rangle$ vectors are not normalized. They are unnormalizable.
- Still, we treat $\langle x|x'\rangle = \delta(x' - x)$ as the correct condition for an orthonormal basis in the continuum case.
- We still have the convenient completeness relation
$$\hat{I} = \int_{-\infty}^{+\infty} \mathrm{d}x\, |x\rangle\langle x|$$

Position Eigenstates and Probabilities

- Note: The $|x\rangle$ state is the eigenstate of the position operator $\hat{x}$ with eigenvalue x,

$$\hat{x}|x\rangle = x|x\rangle$$

- We can take this as the defining property of $|x\rangle$.
- In the discrete case, $\left|\langle\phi_j|\psi\rangle\right|^2$ is the probability of obtaining value a_j in a measurement of $\hat{A}$ when the system is prepared in state $|\psi\rangle$.
- In the continuum, we have

$$|\langle x|\psi\rangle|^2 = \langle\psi|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x' \int_{-\infty}^{+\infty} \mathrm{d}x''\, \psi^*(x')\langle x'|x\rangle\langle x|x''\rangle\psi(x'')$$

$$= \int_{-\infty}^{+\infty} \mathrm{d}x' \int_{-\infty}^{+\infty} \mathrm{d}x''\, \psi^*(x')\delta(x - x')\delta(x'' - x)\psi(x'') = \psi^*(x)\psi(x) = |\psi(x)|^2$$

Position Eigenstates and Probabilities

- Since we are dealing with the continuum, we have to interpret
$$|\langle x|\psi\rangle|^2 = |\psi(x)|^2$$
as the probability *density* for finding the particle at x in a position measurement. In other words
$$\mathrm{Prob}(a \le x \le b) = \int_a^b \mathrm{d}x\, |\psi(x)|^2$$
- In order to have this interpretation, we need
$$\langle\psi|\psi\rangle = \langle\psi|\hat{I}|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, \langle\psi|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi^*(x)\psi(x)$$
$$= \int_{-\infty}^{+\infty} \mathrm{d}x\, |\psi(x)|^2 = 1$$
- Physically realizable states must be normalized $\Rightarrow |x\rangle$ is not a realizable state.

Momentum

- We also want a Hermitian operator representing a particle's momentum. We can proceed as with position and just define an operator

$$\hat{p} = \int_{-\infty}^{+\infty} \mathrm{d}p \, p |p\rangle\langle p|$$

where $|p\rangle$ is a state of definite momentum p and the eigenstates satisfy

$$\langle p'|p\rangle = \delta(p - p')$$

- That works, but we need to know how the $|x\rangle$ and $|p\rangle$ states are related to each other. For that, we actually have to do some physics.

Wave-Particle Duality

- One of the founding ideas of quantum mechanics is that particles sometimes exhibit wave-like behavior and vice versa.
- The *de Broglie hypothesis* states that a free particle of momentum p is associated with a plane wave of wave-number k satisfying

$$p = \hbar k$$

- $\hbar$ is a constant called *Planck's constant*. In this course, we have been implicitly working in units such that $\hbar = 1$, so we'll use $p = k$.
- If the particle has energy E then *Planck's hypothesis* says that the wave has angular frequency ω satisfying

$$E = \hbar\omega$$

or $E = \omega$ in our units.

Momentum In the Position Basis

- The upshot is that we expect the wavefunction of a momentum state to be a plane wave, i.e.

$$|p\rangle = A\int_{-\infty}^{+\infty} \mathrm{d}x\ e^{i(kx-\omega t)}|x\rangle = A\int_{-\infty}^{+\infty} \mathrm{d}x\ e^{i(px-Et)}|x\rangle$$

so that

$$\psi_p(x) = \langle x|p\rangle = Ae^{i(px-Et)}$$

- Now, note that

$$-i\frac{\partial\psi_p(x)}{\partial x} = -iAipe^{i(px-Et)} = p\psi_p(x)$$

- Therefore, if we want $\hat{p}|p\rangle = p|p\rangle$ we need to have

$$\hat{p}|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\ \left(-i\frac{\partial\psi(x)}{\partial x}\right)|x\rangle$$

Momentum In the Position Basis

- The momentum operator $\hat{p}$ maps a wavefunction $\psi(x)$ to $-i\frac{\partial\psi}{\partial x}$.
- We say that the position representation of the momentum operator is

$$\hat{\mathrm{p}} \to -i\frac{\partial}{\partial x}$$

Canonical Commutation Relations

- We can now determine that the position and momentum operators do not commute. In fact
$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hat{I}$$
which is called the *canonical commutation relation*.
- Note: We are often lazy and write $[\hat{x}, \hat{p}] = i$.
- This is responsible for the uncertainty principle: There are no states that predict a precise value for both $\hat{x}$ and $\hat{p}$.
- To derive the commutation relation, we show that
$$[\hat{x}, \hat{p}]|\psi\rangle = i|\psi\rangle$$
for any vector $|\psi\rangle$.

Canonical Commutation Relations

$$[\hat{x},\hat{p}]|\psi\rangle = (\hat{x}\hat{p} - \hat{p}\hat{x}) \int_{-\infty}^{+\infty} \mathrm{d}x \, \psi(x)|x\rangle$$

$$= \int_{-\infty}^{+\infty} \mathrm{d}x \, [\hat{x}(\hat{p}\psi(x)|x\rangle) - \hat{p}(\hat{x}\psi(x)|x\rangle)]$$

$$= \int_{-\infty}^{+\infty} \mathrm{d}x \left[x\left(-i\frac{\partial\psi}{\partial x}\right) - \left(-i\frac{\partial}{\partial x}\left(x\psi(x)\right)\right)\right]|x\rangle$$

$$= i\int_{-\infty}^{+\infty} \mathrm{d}x \left[-x\frac{\partial\psi}{\partial x} + \psi(x) + x\frac{\partial\psi}{\partial x}\right]|x\rangle$$

$$= i\int_{-\infty}^{+\infty} \mathrm{d}x \, \psi(x)|x\rangle = i|\psi\rangle$$

Functions of Operators

- Suppose a function $f(t)$ has a Taylor series
$$f(t) = \sum_{n=0}^{\infty} a_n t^n$$
- Then, for an operator $\hat{A}$, we define
$$f(\hat{A}) = \sum_{n=0}^{\infty} a_n \hat{A}^n$$
- In particular, $e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{A}^n$
- From this you can derive that, if $\hat{A}$ and $\hat{B}$ commute then
$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}}$$

Formal Solution of the Schrödinger Equation

- Note that, if $\hat{A}$ is Hermitian, then $\hat{U} = e^{i\hat{A}}$ is unitary
$$\hat{U}^\dagger \hat{U} = e^{-i\hat{A}^\dagger} e^{i\hat{A}} = e^{-i\hat{A}} e^{i\hat{A}} = e^{-i\hat{A}+i\hat{A}} = e^{i0} = \hat{I}$$
- We know that discrete time dynamics is unitary
$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle$$
- and that continuous time dynamics satisfies the Schrödinger equation
$$i\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle$$
$$\Rightarrow \quad i\frac{\partial\hat{U}(t, t_0)}{\partial t}|\psi(t_0)\rangle = \hat{H}\hat{U}(t, t_0)|\psi(t_0)\rangle$$
- Because this has to hold for any initial state $|\psi(t_0)\rangle$, we have
$$i\frac{\partial\hat{U}(t, t_0)}{\partial t} = \hat{H}\hat{U}(t, t_0)$$

Formal Solution of the Schrödinger Equation

- If $\hat{H}$ is independent of time, then the solution to this equation is

$$\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)}$$

- Check:

$$i\frac{\partial \hat{U}(t, t_0)}{\partial t} = i\left(-i\hat{H}e^{-i\hat{H}(t-t_0)}\right) = \hat{H}e^{-i\hat{H}(t-t_0)} = \hat{H}\hat{U}(t, t_0)$$

- Note, we want $\hat{U}(t_0, t_0) = \hat{I}$, which is why we must have $(t - t_0)$ in the exponential rather than $(t + a)$ for an arbitrary a.
- The operator $\hat{U}(t, t_0)$ is called the *propagator*.

Translation Operators

- An operator of the form $\hat{U}(a) = e^{-ia\hat{p}}$ is called a *translation operator.*

- $\hat{U}(a)$ is unitary because $-a\hat{p}$ is Hermitian.

- Next, consider $\hat{U}(\Delta a)\hat{x}\hat{U}^\dagger(\Delta a)$ for small Δa

$$\begin{aligned}
\hat{U}(\Delta a)\hat{x}\hat{U}^\dagger(\Delta a) &= \left(\hat{I} - i\Delta a\hat{p}\right)\hat{x}\left(\hat{I} + i\Delta a\hat{p}\right) + O(\Delta a^2) \\
&= \hat{x} + i\Delta a(\hat{x}\hat{p} - \hat{p}\hat{x}) + O(\Delta a^2) \\
&= \hat{x} + i\Delta a[\hat{x}, \hat{p}] \\
&= \hat{x} + i\Delta a(i\hat{I}) \\
&= \hat{x} - \Delta a\hat{I}
\end{aligned}$$

Translation Operators

- From this, we can derive

$$\hat{U}(a)\hat{x}\hat{U}^\dagger(a) = \hat{x} - a\hat{I}$$

$$\begin{aligned}
\hat{U}(a)\hat{x}\hat{U}^\dagger(a) &= \lim_{N\to\infty}\left(\left[\hat{U}\left(\frac{a}{N}\right)\right]^N \hat{x}\left[\hat{U}^\dagger\left(\frac{a}{N}\right)\right]^N\right)\\
&= \lim_{N\to\infty}\left(\left[\hat{U}\left(\frac{a}{N}\right)\right]^{N-1}\left(\hat{x} - \frac{a}{N}\hat{I}\right)\left[\hat{U}^\dagger\left(\frac{a}{N}\right)\right]^{N-1} + O\left(\frac{1}{N^2}\right)\right)\\
&= \lim_{N\to\infty}\left(\left[\hat{U}\left(\frac{a}{N}\right)\right]^{N-1}\hat{x}\left[\hat{U}^\dagger\left(\frac{a}{N}\right)\right]^{N-1} - \frac{a}{N}\hat{I} + O\left(\frac{1}{N^2}\right)\right)\\
&= \lim_{N\to\infty}\left(\hat{x} - a\hat{I} + O\left(\frac{1}{N}\right)\right) = \hat{x} - a\hat{I}
\end{aligned}$$

Translation Operators

- Further, we can derive
$$\hat{U}(a)|x\rangle = |x+a\rangle$$
- Start with the eigenvalue equation and act with $\hat{U}(a)$

$$\hat{x}|x\rangle = x|x\rangle$$
$$\hat{U}(a)\hat{x}|x\rangle = x\hat{U}(a)|x\rangle$$
$$\hat{U}(a)\hat{x}\hat{U}^\dagger(a)\hat{U}(a)|x\rangle = x\hat{U}(a)|x\rangle \qquad \text{by unitatiry}$$
$$\left(\hat{x} - a\hat{I}\right)\left(\hat{U}(a)|x\rangle\right) = x\left(\hat{U}(a)|x\rangle\right)$$
$$\hat{x}\left(\hat{U}(a)|x\rangle\right) = (x+a)\left(\hat{U}(a)|x\rangle\right)$$

- In other words, $\hat{U}(a)|x\rangle$ is an eigenstate of $\hat{x}$ with eigenvalue $x + a$, which is precisely the definition of $|x+a\rangle$.

Translation Operators

- We can now see what $\hat{U}(a)$ does to the wavefunction $\psi(x)$

$$\hat{U}(a)|\psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x)\hat{U}(a)|x\rangle$$
$$= \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x)|x+a\rangle$$
$$= \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x-a)|x\rangle$$

- The wavefunction $\psi'(x) = \langle x|\hat{U}(a)|\psi\rangle = \psi(x-a)$ of $\hat{U}(a)|\psi\rangle$ is the wavefunction of $|\psi\rangle$, translated to the right by a. Hence the name *translation operator*.

Translation Hamiltonian

- Suppose now that the Hamiltonian of our system is proportional to the momentum

$$\hat{H} = g\hat{p}$$

- The propagator $\hat{U}(t, t_0) = e^{-ig(t-t_0)\hat{p}}$ is a translation operator, so the wavefunction will move to the right at a rate g.