Quantum Foundations Lecture 21

April 23, 2018
Dr. Matthew Leifer
leifer@chapman.edu
HSC112

Announcements

- Assignments: Final Version due May 2.
- Homework 4 due April 30.

The PBR Theorem

- The PBR Theorem (Nature Physics 8:475-478 (2012)) proves that ontological models have to be ψ-ontic under an additional assumption called the Preparation Independence Postulate (PIP).

- The PIP can be broken down into two assumptions:

  - The Cartesian Product Assumption:

  When two systems are prepared independently in a product state $|\psi\rangle_A \otimes |\phi\rangle_B$, the joint ontic state space is $\Lambda_{AB} = \Lambda_A \times \Lambda_B$, i.e. each system has its own ontic state, i.e. the ontic state of the joint system is $\lambda_{AB} = (\lambda_A, \lambda_B)$, where λ_A is the ontic state of system A and λ_B is the ontic state of system B.

  - The No Correlation Assumption:

  The epistemic state corresponding to $|\psi\rangle_A \otimes |\phi\rangle_B$ is:

$$\Pr(\lambda_A, \lambda_B | \psi_A, \phi_B) = \Pr(\lambda_A | \psi_A)\Pr(\lambda_B | \phi_B)$$

Comments on the PIP

- In general, a joint system with two subsystems might have global ontic properties that do not reduce to properties of the individual subsystems.
    - In a ψ-ontic model with entangled states this would be the case: $|\psi\rangle_{AB}$ is not a property of either subsystem.
    - So, in general, we need $\Lambda_{AB} = \Lambda_A \times \Lambda_B \times \Lambda_{\text{global}}$.
    - All we really require from the Cartesian Product Assumption is that Λ_{global} plays no role in determining measurement outcomes when we prepare a product state, e.g. for product states $\lambda_g \in \Lambda_{\text{global}}$ always takes the same specific value.
    - Then, the No Correlation Assumption should be read as applying to the marginal on $\Lambda_A \times \Lambda_B$.

The PBR Theorem

- **Theorem**: An ontological model of quantum theory that satisfies the PIP must be ψ-ontic.

- Proof strategy: We follow a proof by C. Moseley (arXiv:1401.0026)

  1. Prove that $|\psi_1\rangle$ and $|\psi_2\rangle$ must be ontologically distinct whenever $|\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{2}$ using *antidistinguishability*.

  2. Prove the case $|\langle\psi_1|\psi_2\rangle|^2 < \frac{1}{2}$ by reduction to 1.

  3. Prove the case $\frac{1}{2} < |\langle\psi_1|\psi_2\rangle|^2 < 1$ by reduction to 2.

Antidistinguishability

- A set $\{\rho_j\}_{j=1}^{n}$ of n quantum states is *antidistinguishable* if there exists an n-outcome POVM $\{E_j\}_{j=1}^{n}$ such that

$$\forall j, \qquad \text{Tr}(E_j \rho_j) = 0$$

- Example:
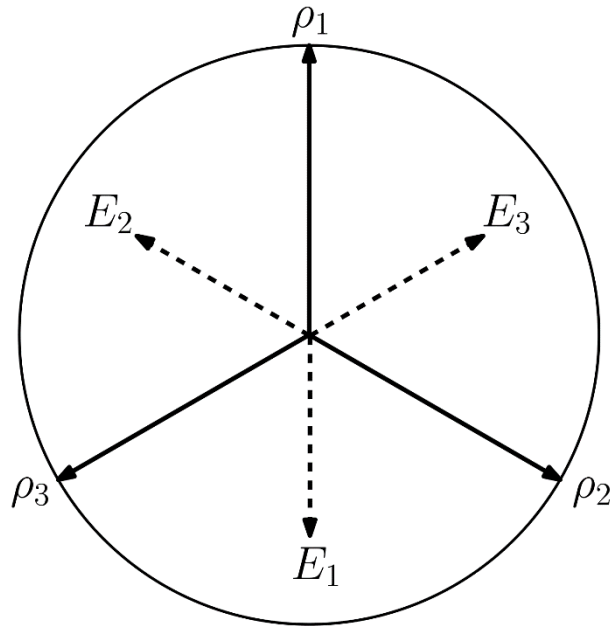
Antidistinguishability

- We define the n-way overlap as
$$L_e(\rho_1, \rho_2, \cdots, \rho_n) = \int_\Lambda \min_j \{\Pr(\lambda|\rho_j)\} \, d\lambda$$

- Lemma: If a set of states is antidistinguishable, then, in any ontological model $L_e(\rho_1, \rho_2, \ldots, \rho_n) = 0$.

- Proof for finite Λ:
  - $L_e(\rho_1, \rho_2, \cdots, \rho_n) = \sum_\lambda \min_j \{\Pr(\lambda|\rho_j)\}$ so it is > 0 iff there exists a λ for which all $\Pr(\lambda|\rho_j) > 0$.
  - Suppose there exists such a λ. We require $\Pr(E_j|\lambda) = 0$ for all E_j in order to reproduce the quantum predictions.
  - But $\sum_{j=1}^n \Pr(E_j|\lambda) = 1$, so no such λ can exist.

The Real Proof

- By antidistinguishability

$$0 = \sum_{k=1}^{n} \mathrm{Tr}(E_k \rho_k)$$

$$= \int_{\Lambda} \left[\sum_{k} \mathrm{Pr}(E_k|\lambda)\mathrm{Pr}(\lambda|\rho_k) \right] d\lambda$$

$$\geq \int_{\Lambda} \left[\sum_{k} \mathrm{Pr}(E_k|\lambda) \min_{j}\{\mathrm{Pr}(\lambda|\rho_j)\} \right] d\lambda$$

$$= \int_{\Lambda} \left[\sum_{k} \mathrm{Pr}(E_k|\lambda) \right] \min_{j}\{\mathrm{Pr}(\lambda|\rho_j)\} d\lambda$$

- But $\sum_{k=1}^{n} \mathrm{Pr}(E_k|\lambda) = 1$, so

$$0 = \int_{\Lambda} \min_{j}\{\mathrm{Pr}(\lambda|\rho_j)\} d\lambda = L_e(\rho_1, \rho_2, \cdots, \rho_n)$$

Case: $|\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{2}$

⊙ Without loss of generality, we can choose a basis such that

$$|\psi_1\rangle = |0\rangle, \qquad |\psi_2\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

⊙ Now consider the four states

$$|\Psi_1\rangle_{AB} = |0\rangle_A \otimes |0\rangle_B, \qquad |\Psi_2\rangle_{AB} = |0\rangle_A \otimes |+\rangle_B$$
$$|\Psi_3\rangle_{AB} = |+\rangle_A \otimes |0\rangle_B, \qquad |\Psi_4\rangle_{AB} = |+\rangle_A \otimes |+\rangle_B$$

and the orthonormal basis

$$|\Phi_1\rangle_{AB} = \tfrac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$$

$$|\Phi_2\rangle_{AB} = \tfrac{1}{\sqrt{2}}(|0\rangle_A \otimes |-\rangle_B + |1\rangle_A \otimes |+\rangle_B)$$

$$|\Phi_3\rangle_{AB} = \tfrac{1}{\sqrt{2}}(|+\rangle_A \otimes |1\rangle_B + |-\rangle_A \otimes |0\rangle_B)$$

$$|\Phi_4\rangle_{AB} = \tfrac{1}{\sqrt{2}}(|+\rangle_A \otimes |-\rangle_B + |-\rangle_A \otimes |+\rangle_B)$$

where $|\pm\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

⊙ You can verify that $\langle\Phi_j|\Psi_j\rangle = 0$, so a measurement in this basis so $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle\}$ is antidistinguishable.

Case: $|\langle \psi_1 | \psi_2 \rangle|^2 = \frac{1}{2}$
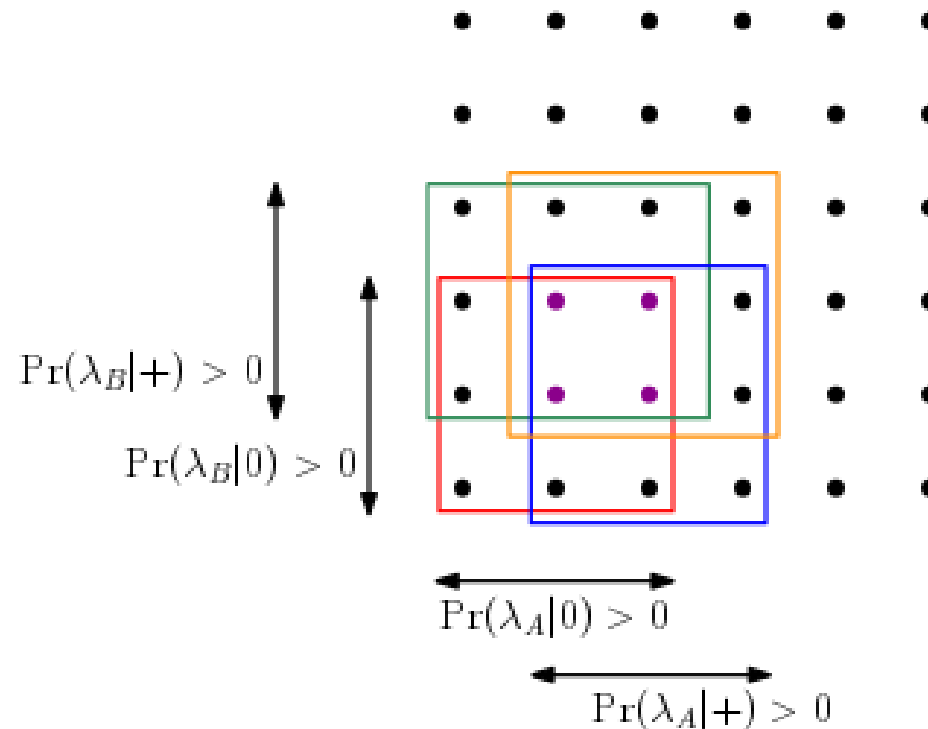
⊙ By the PIP:

$Pr(\lambda|\psi_1) = Pr(\lambda_A|0)Pr(\lambda_B|0)$
$Pr(\lambda|\psi_2) = Pr(\lambda_A|0)Pr(\lambda_B|+)$
$Pr(\lambda|\psi_3) = Pr(\lambda_A|+)Pr(\lambda_B|0)$
$Pr(\lambda|\psi_4) = Pr(\lambda_A|+)Pr(\lambda_B|+)$

⊙ For finite Λ, in order to avoid having the purple ontic states $Pr(\lambda|0)$ and $Pr(\lambda|+)$ must have no overlap.

$Pr(\lambda_A, \lambda_B|\Psi_1) > 0$
$Pr(\lambda_A, \lambda_B|\Psi_2) > 0$
$Pr(\lambda_A, \lambda_B|\Psi_3) > 0$
$Pr(\lambda_A, \lambda_B|\Psi_4) > 0$

● = ruled out by antidistinguishability

Case: $|\langle \psi_1 | \psi_2 \rangle|^2 = \frac{1}{2}$

⊙ General proof:

$$0 = L_e(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = \int_{\Lambda_A} d\lambda_A \int_{\Lambda_B} d\lambda_B \left[\min_j \{ \Pr(\lambda_A, \lambda_B | \Psi_j) \} \right]$$

$$= \int_{\Lambda_A} d\lambda_A \int_{\Lambda_B} d\lambda_B \left[\min \left\{ \begin{matrix} \Pr(\lambda_A|0)\Pr(\lambda_B|0), \Pr(\lambda_A|0)\Pr(\lambda_B|+), \Pr(\lambda_A|+)\Pr(\lambda_B|0), \\ \Pr(\lambda_A|+)\Pr(\lambda_B|+) \end{matrix} \right\} \right]$$

$$= \left[\int_{\Lambda_A} d\lambda_A \min\{\Pr(\lambda_A|0), \Pr(\lambda_A|+)\} \right] \times \left[\int_{\Lambda_B} d\lambda_B \min\{\Pr(\lambda_B|0), \Pr(\lambda_B|+)\} \right]$$

$$= L_e(0, +)^2$$

Case: $|\langle\psi_1|\psi_2\rangle|^2 < \frac{1}{2}$

- **Theorem**: If $|\langle\psi_1|\psi_2\rangle|^2 < |\langle\psi_3|\psi_4\rangle|^2$ then there exists a CPT map $\mathcal{E}$ such that

$$\mathcal{E}(|\psi_1\rangle\langle\psi_1|) = |\psi_3\rangle\langle\psi_3|, \qquad \mathcal{E}(|\psi_2\rangle\langle\psi_2|) = |\psi_4\rangle\langle\psi_4|$$

- So, our measurement procedure will consist of mapping $|\psi_1\rangle$ to $|0\rangle$, $|\psi_2\rangle$ to $|+\rangle$, and then applying the same measurement as before.

- We can always choose a basis such that

$$|\psi_1\rangle = |0\rangle, \qquad |\psi_2\rangle = \sin\theta|0\rangle + \cos\theta|1\rangle,$$

with $0 \le \theta < \frac{\pi}{4}$.

- Then, you can check that $\mathcal{E}(\rho) = M_1\rho M_1^\dagger + M_2\rho M_2^\dagger$ with

$$M_1 = |0\rangle\langle0|+\tan\theta|1\rangle\langle1|, \qquad M_2 = \sqrt{\frac{1 - \tan^2\theta}{2}}(|0\rangle + |1\rangle)\langle1|$$

is CPT and does the job.

Case: $\frac{1}{2} < |\langle\psi_1|\psi_2\rangle|^2 < 1$

- Let
$$|\Psi_1\rangle = |\psi_1\rangle^{\otimes n} \text{ and } |\Psi_2\rangle = |\psi_2\rangle^{\otimes n}$$

- Since $|\langle\Psi_1|\Psi_2\rangle|^2 = |\langle\psi_1|\psi_2\rangle|^{2n}$, there exists an n such that
$$|\langle\Psi_1|\Psi_2\rangle|^2 < \frac{1}{2}$$

- Apply the previous argument to $|\Psi_1\rangle$ and $|\Psi_2\rangle$, i.e. the four states
$$|\Psi_1\rangle \otimes |\Psi_1\rangle, \qquad |\Psi_1\rangle \otimes |\Psi_2\rangle$$
$$|\Psi_2\rangle \otimes |\Psi_1\rangle, \qquad |\Psi_2\rangle \otimes |\Psi_2\rangle$$

- From this, we get $L_e(\Psi_1, \Psi_2) = 0$

Case: $\frac{1}{2} < |\langle\psi_1|\psi_2\rangle|^2 < 1$

⊙ By the PIP,

$$\Pr(\lambda_1, \lambda_2, \cdots, \lambda_n|\Psi_1) = \Pr(\lambda_1|\psi_1)\Pr(\lambda_2|\psi_1)\cdots\Pr(\lambda_n|\psi_1)$$
$$\Pr(\lambda_1, \lambda_2, \cdots, \lambda_n|\Psi_2) = \Pr(\lambda_1|\psi_2)\Pr(\lambda_2|\psi_2)\cdots\Pr(\lambda_n|\psi_2)$$

and hence

$$0 = L_e(\Psi_1, \Psi_2)$$

$$= \int_{\Lambda_1} d\lambda_1 \int_{\Lambda_2} d\lambda_2 \cdots \int_{\Lambda_n} d\lambda_n \min\{\Pr(\lambda_1, \lambda_2, \cdots, \lambda_n|\Psi_1), \Pr(\lambda_1, \lambda_2, \cdots, \lambda_n|\Psi_2)\}$$

$$= \int_{\Lambda_1} d\lambda_1 \min\{\Pr(\lambda_1|\psi_1), \Pr(\lambda_1|\psi_1)\} \times \int_{\Lambda_2} d\lambda_2 \min\{\Pr(\lambda_2|\psi_1), \Pr(\lambda_2|\psi_1)\} \times \cdots$$

$$\times \int_{\Lambda_n} d\lambda_n \min\{\Pr(\lambda_n|\psi_1), \Pr(\lambda_n|\psi_1)\}$$

$$= \left[\int_{\Lambda} d\lambda \min\{\Pr(\lambda|\psi_1), \Pr(\lambda|\psi_2)\}\right]^n = L_e(\psi_1, \psi_2)^n$$

Prospects for ψ-ontology theorems

- The PBR theorem renders ψ-epistemic explanations implausible within the ontological models framework.
- The upshot of overlap bounds is more ambiguous.
- Apart from fundamental interest, ψ-ontology theorems are interesting because they imply most of the other known no-go theorems.
  - From this point of view, the extra assumptions needed for PBR are not ideal.
- It is still possible that:
  - Better overlap bounds could be obtained.
  - ψ-epistemic models are impossible for infinite dimensional Hilbert Spaces.
  - ψ-epistemic models are impossible for POVMs (We already know that the Kochen-Specker model cannot be extended to POVMs).

9.vii) Bell's Theorem

- Recall, back in lecture 10, we described the EPR argument.
- The entangled state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

  exhibits perfect correlations when both Alice and Bob measure in the $\{|0\rangle, |1\rangle\}$ basis (or indeed the $\{|\vec{n}+\rangle, |\vec{n}-\rangle\}$ basis with $\vec{n}$ in the $x - z$ plane.

- According to the orthodox interpretation, Bob's outcomes "pop into existence" nonlocally when Alice makes her measurement and the quantum state collapses.

- EPR argued that the measurement outcomes must pre-exist in order to avoid nonlocality.

- This is exactly how it works in the Spekkens toy theory.
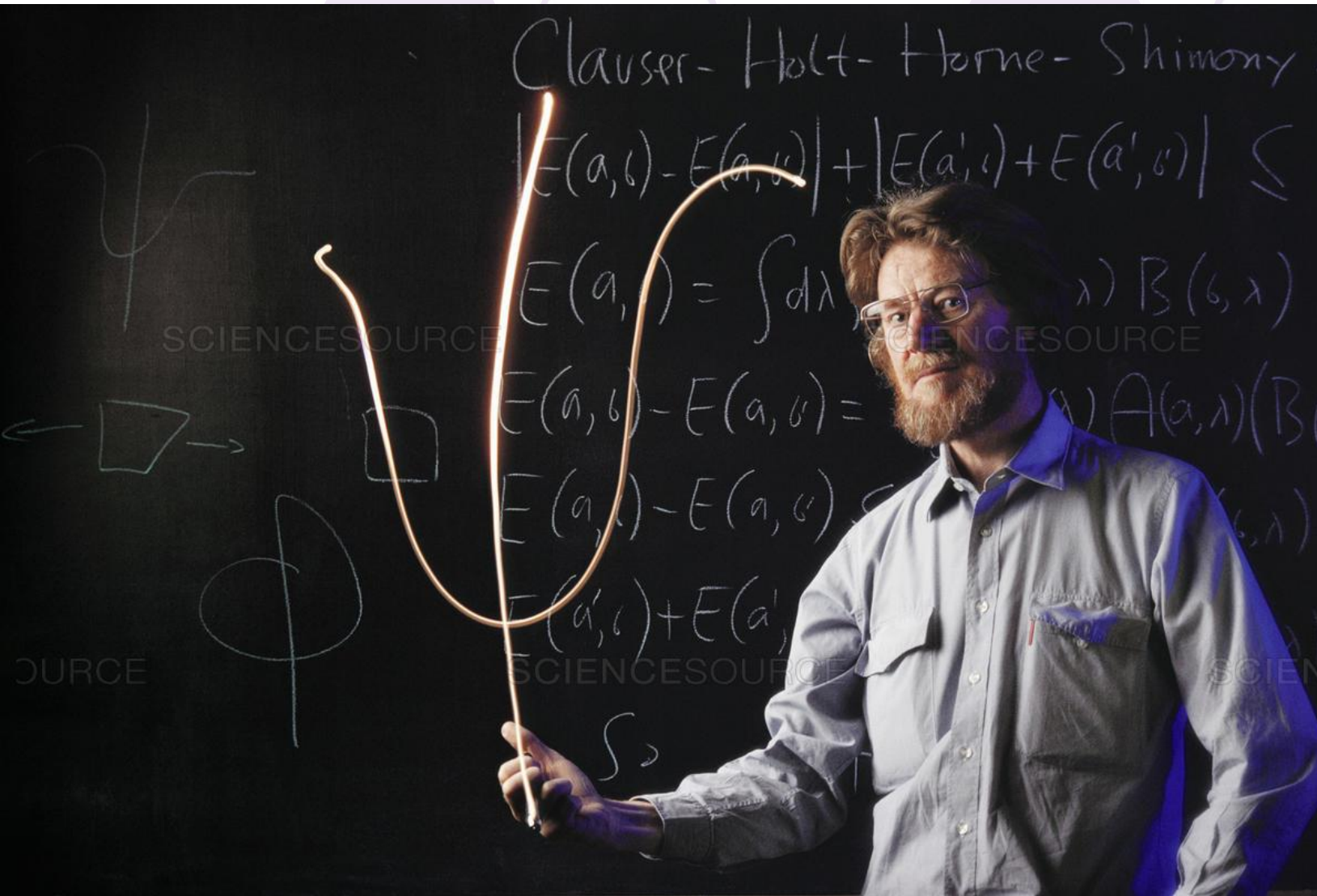
- In 1964, John Stewart Bell proved that the correlations of entangled quantum systems cannot be explained in this way.

- We will explain a version due to Clauser, Horne, Shimony and Holt.

The CHSH Game

- Get into groups of four.
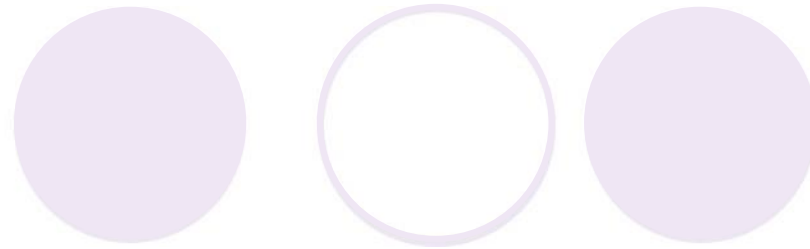 - In each group, choose one person to be:

Alice

Bob

Charlie

Dora

The CHSH Game

1. Alice and Bob get together for a few minutes to decide their strategy.
2. Bob leaves the room with Dora. Alice and Charlie stay.
3. Charlie and Dora each flip a coin. Write down the outcome.
4. Alice and Bob have to write either +1 or -1 in response.
5. Alice and Bob win the game if:
   - Whenever the coin flips are HH, HT, or TH, they give the same answer.
   - Whenever the coin flips are TT, they give a different answer.
6. Repeat steps 3-5.
7. Bob and Dora come back in the room. They count how many times Alice and Bob won as well as the total number of times they played.
8. Report the results back to me. The goal is to win the game 85% of the time.

Why can't Alice and Bob always win?
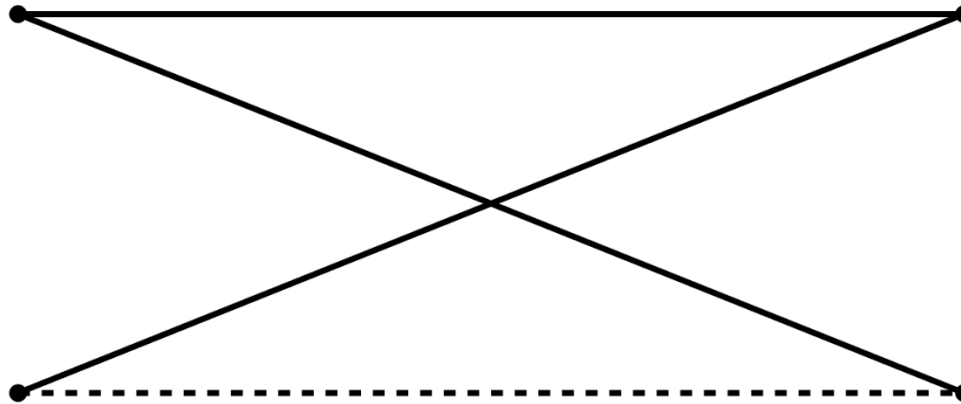

Charlie's coin flip — H

Alice's answer

Bob's answer

Dora's coin flip — H

T

T

- Assuming the coin flips are uniformly random, Alice and Bob will win at most 75% of the time in the long run.

List of All Deterministic Strategies

Alice	H	+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-
	T	+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-
Bob	H	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-
	T	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-
Winning Probability		75%	75%	25%	25%	75%	25%	75%	25%	25%	75%	25%	75%	25%	25%	75%	75%

Do Nondterministic Strategies Matter?

- Suppose Alice and Bob do not choose a fixed strategy, but use classical randomness (coin flips, dice throws, etc.) to choose it each time, i.e. they decide to use strategy j, k, l, m with probability $p_{j,k,l,m}$ $(j, k, l, m = \pm 1)$.

- On each round of the game they will still end up using a deterministic strategy with winning probability $\leq 75\%$.

- The average of the winning probability cannot be higher than the winning probability for the best deterministic strategy.

- Alice and Bob might as well just pick the best deterministic strategy.

What About Delaying the Decision?

- Alice and Bob each have four local deterministic strategies (ignoring what the other person is doing)

H	+	+	-	-
T	+	-	+	-

- Alice and Bob could decide as follows:
  - Alice waits until she sees the outcome of her coin flip.
  - If it is H, she picks $+/-$ with probability $p_\pm^H$ (e.g. by flipping a biased coin)
  - If it is T, she picks $+/-$ with probability $p_\pm^T$
  - Bob does similarly with distributions $q_\pm^H$ and $q_\pm^T$
- But this just amounts to picking strategy j, k, l, m with probability
$$p_{j,k,l,m} = p_j^H p_k^T q_l^H q_m^H$$
- In other words, Alice and Bob could just have flipped all their coins at the beginning, so we are back to the previous case.
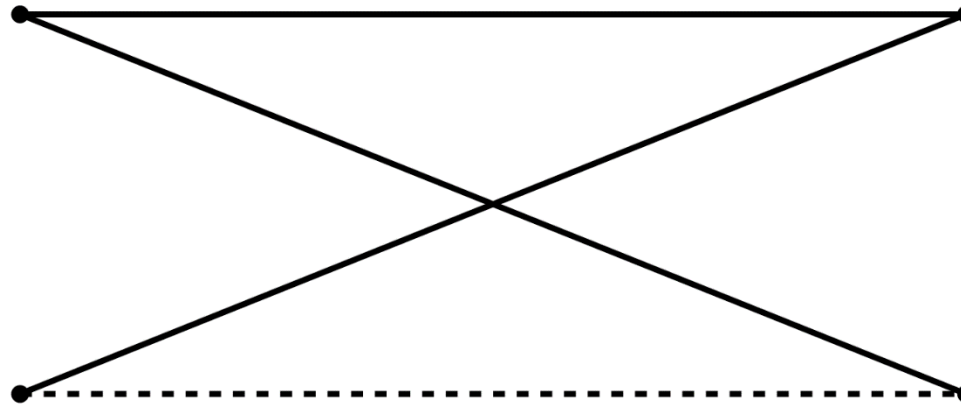
The CHSH Inequality

Charlie's coin flip | Alice's answer | Bob's answer | Dora's coin flip

H | | | H

T | | | T

- Whatever strategy Alice and Bob use (deterministic, nodeterministic, delayed), their outcome probabilities satisfy

$$P(a = b|H,H) + P(a = b|H,T) + P(a = b|T,H) + P(a \neq b|TT) \leq 3$$

- This is (a version of) the CHSH inequality.

Quantum Violation

- Now suppose that we allow Alice and Bob to use quantum systems to play the game.

- They initially prepare two qubits in a state $|\psi\rangle_{AB}$. Alice takes system A with her and Bob takes system B.

- If Alice's coin flip is heads, she measures her system in the basis $\{|\vec{n}_{\mathrm{H}}+\rangle, |\vec{n}_{\mathrm{H}}-\rangle\}$. If she gets the $|\vec{n}_{\mathrm{H}} \pm\rangle$ outcome she answers $a = \pm 1$.

- If Alice's coin flip is tails, she measures her system in the basis $\{|\vec{n}_{\mathrm{T}}+\rangle, |\vec{n}_{\mathrm{T}}-\rangle\}$. If she gets the $|\vec{n}_{\mathrm{T}} \pm\rangle$ outcome she answers $a = \pm 1$.

- Bob does the same thing on his system with the bases $\{|\vec{m}_{\mathrm{H}}+\rangle, |\vec{m}_{\mathrm{H}}-\rangle\}$ and $\{|\vec{m}_{\mathrm{T}}+\rangle, |\vec{m}_{\mathrm{T}}-\rangle\}$.

Quantum Violation

- Suppose Alice and Bob prepare the state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$$

- Fortunately, you proved in Hwk 3, that the outcome probabilities are

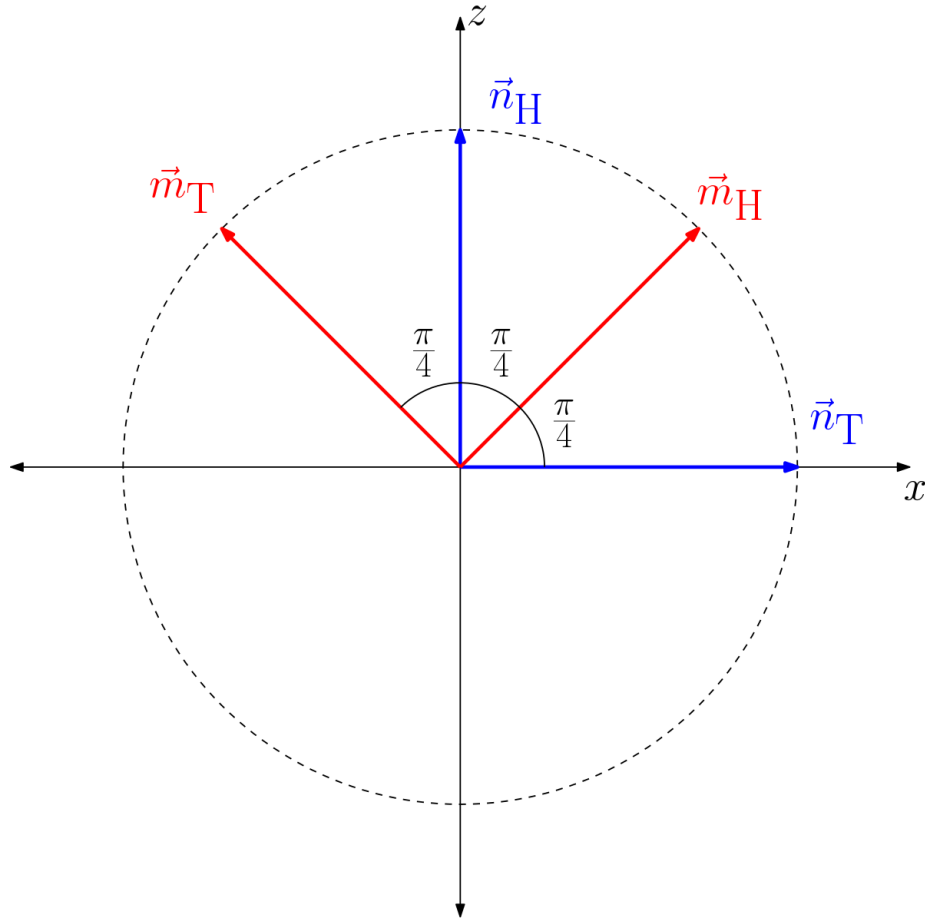$$P(\vec{n}+, \vec{m}\,+) = \tfrac{1}{2}\cos^2\tfrac{\phi}{2}, \qquad P(\vec{n}+, \vec{m}\,-) = \tfrac{1}{2}\sin^2\tfrac{\phi}{2}$$

$$P(\vec{n}-, \vec{m}\,-) = \tfrac{1}{2}\cos^2\tfrac{\phi}{2}, \qquad P(\vec{n}-, \vec{m}\,+) = \tfrac{1}{2}\sin^2\tfrac{\phi}{2}$$

  where ϕ is the angle between $\vec{n}$ and $\vec{m}$ on the x-z plane of the Bloch sphere.

- So we just have to choose the measurement angles and see what we get.
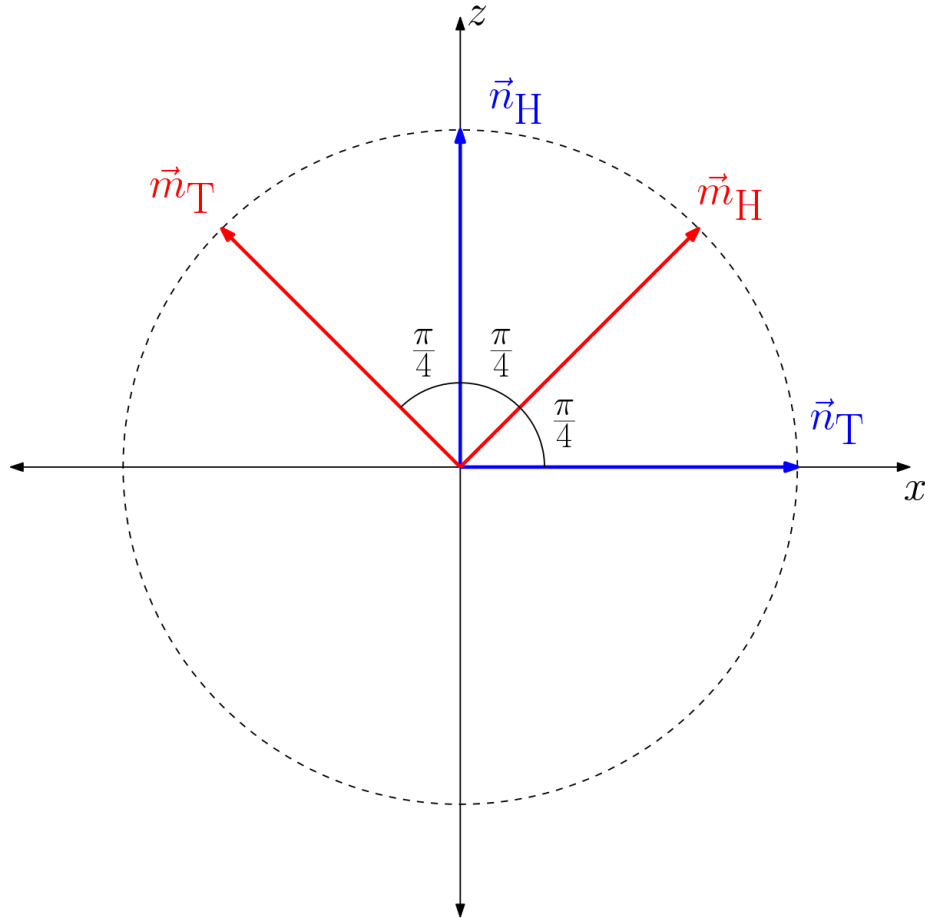
Quantum Violation

$$P(a = b | \mathrm{H}, \mathrm{H})$$

$$= P(\vec{n}_\mathrm{H}+, \vec{m}_\mathrm{H}+) + P(\vec{n}_\mathrm{H}-, \vec{m}_\mathrm{H}-)$$

$$= \cos^2\left(\frac{\pi}{8}\right)$$

$$= \frac{1 + \cos\left(\frac{\pi}{4}\right)}{2}$$

$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$
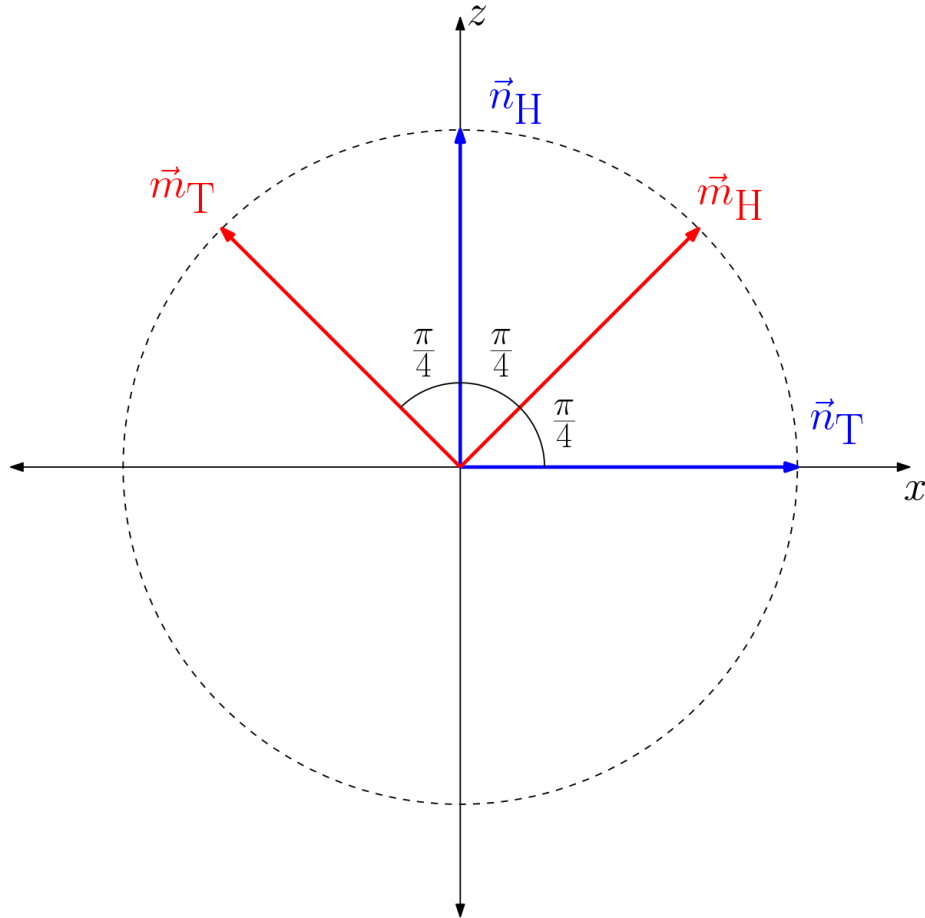
Quantum Violation

$$P(a = b | \text{H}, \text{T})$$

$$= P(\vec{n}_{\text{H}}+, \overrightarrow{m_{\text{T}}} +) + P(\vec{n}_{\text{H}}-, \overrightarrow{m_{\text{T}}} -)$$

$$= \cos^2\left(\frac{\pi}{8}\right)$$

$$= \frac{1 + \cos\left(\frac{\pi}{4}\right)}{2}$$

$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$
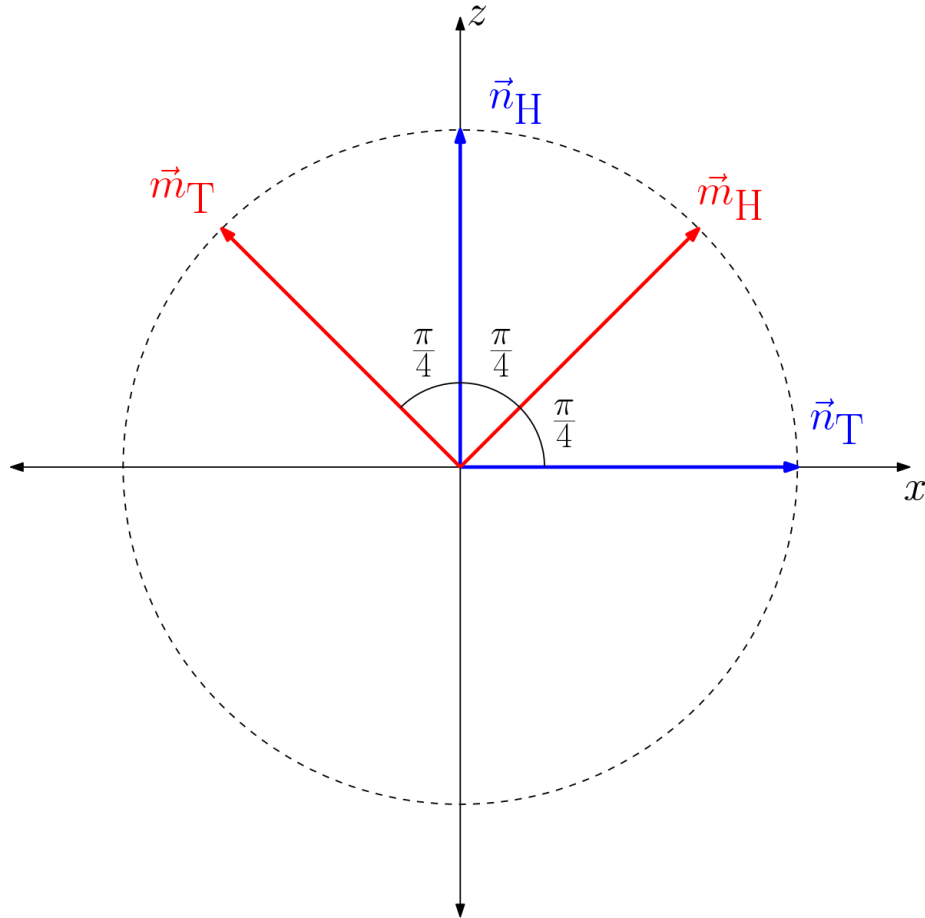
Quantum Violation

$$P(a = b | \text{T}, \text{H})$$

$$= P(\vec{n}_\text{T}+, \vec{m}_\text{H}+) + P(\vec{n}_\text{T}-, \vec{m}_\text{H}-)$$

$$= \cos^2\left(\frac{\pi}{8}\right)$$

$$= \frac{1 + \cos\left(\frac{\pi}{4}\right)}{2}$$

$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

Quantum Violation

$$P(a \neq b | \mathrm{T}, \mathrm{T})$$

$$= P(\vec{n}_\mathrm{T}+, \vec{m}_\mathrm{T}-) + P(\vec{n}_\mathrm{T}-, \vec{m}_\mathrm{T}+)$$

$$= \sin^2\left(\frac{3\pi}{8}\right) = \cos^2\left(\frac{\pi}{8}\right)$$

$$= \frac{1 + \cos\left(\frac{\pi}{4}\right)}{2}$$

$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

Quantum Violation

- Therefore, in the quantum case, we can get

$$P(a = b|H, H) + P(a = b|H, T) + P(a = b|T, H) + P(a \neq b|TT)$$

$$4\cos^2\left(\frac{\pi}{8}\right) = 2\left(1 + \frac{1}{\sqrt{2}}\right) \approx 3.141 > 3$$
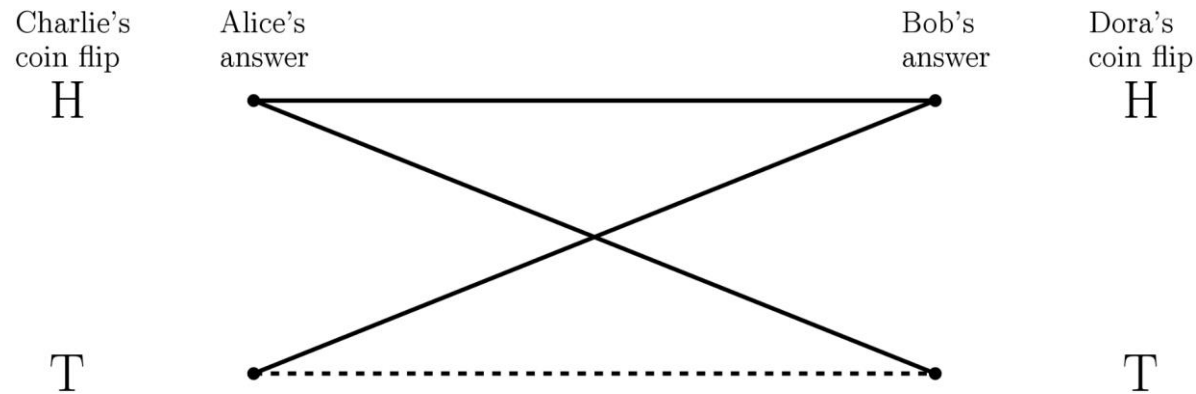
- Therefore, with quantum mechanics you can win the game with probability

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \approx 85.4\% > 75\%$$

- This is actually the maximum possible success probability in quantum mechanics, known as the *Tsirelson bound*.
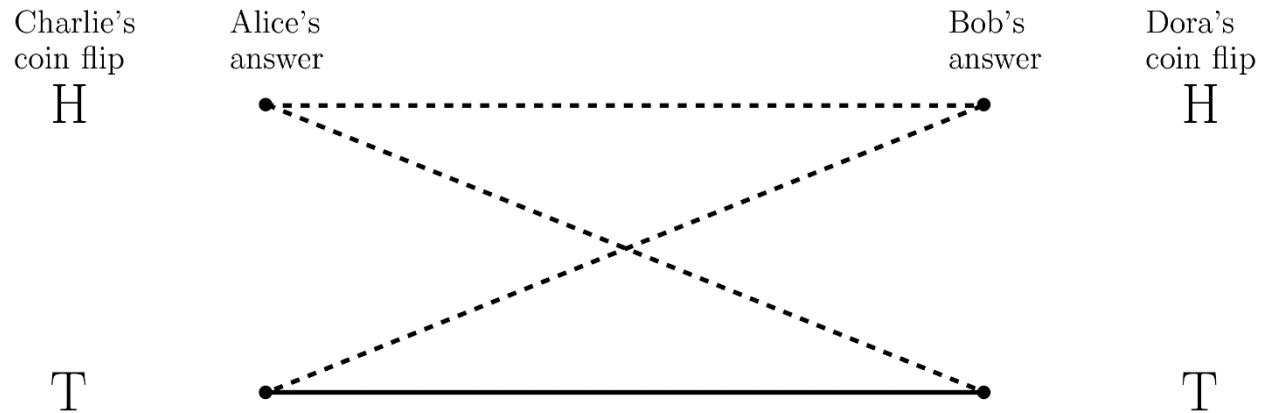
The Usual Form of the CHSH Inequality

- The CHSH inequality is usually expressed in terms of expectation values of observables rather than probabilities.
- To do this, note that we actually have four inequalities

$$1 \leq P(a = b|H, H) + P(a = b|H, T) + P(a = b|T, H) + P(a \neq b|TT) \leq 3$$

The Usual Form of the CHSH Inequality

Charlie's coin flip
Alice's answer
Bob's answer
Dora's coin flip

$$1 \leq P(a \neq b | H, H) + P(a \neq b | H, T) + P(a \neq b | T, H) + P(a = b | TT) \leq 3$$

or

$$-3 \leq -P(a \neq b | H, H) - P(a \neq b | H, T) - P(a \neq b | T, H) - P(a = b | TT) \leq -1$$

The Usual Form of the CHSH Inequality

- Because Alice and Bob's answers a, b take values ± 1

$$\langle ab \rangle = P(a = b) - P(a \neq b)$$

$$1 \leq P(a = b|H,H) + P(a = b|H,T) + P(a = b|T,H) + P(a \neq b|TT) \leq 3$$
$$-3 \leq -P(a \neq b|H,H) - P(a \neq b|H,T) - P(a \neq b|T,H) - P(a = b|TT) \leq -1$$

- Summing these gives:

$$-2 \leq \langle ab \rangle_{\mathrm{HH}} + \langle ab \rangle_{\mathrm{HT}} + \langle ab \rangle_{\mathrm{TH}} - \langle ab \rangle_{\mathrm{TT}} \leq 2$$

which is the usual CHSH inequality.

- And our quantum strategy gives

$$4\cos^2\left(\frac{\pi}{8}\right) - 4\left(1 - \cos^2\left(\frac{\pi}{8}\right)\right) = 8\cos^2\left(\frac{\pi}{8}\right) - 4 = \frac{8}{2}\left(1 + \frac{1}{\sqrt{2}}\right) - 4 = 2\sqrt{2} \approx 2.828$$

which is what is usually called the Tsirelson bound.