Quantum Foundations Lecture 20

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HSC112

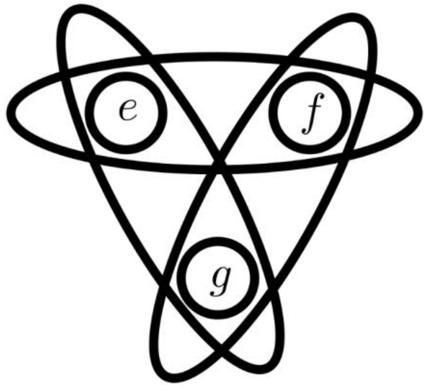
Announcements

- Emergency Phyzza: Monday 4/23 AF207.
- Assignments: Final Version due May 2.
- Homework 4 due April 25.

KS Contextuality in Test Spaces

- The 18-ray proof is based on a test space. We can generalize this approach to arbitrary test spaces.
- Recall that a *finite test space* (X, Σ) consists of
  - A finite set X of *outcomes*.
  - A finite set Σ of *tests*.
  - Each test E is a finite subset of X, interpreted as the set of outcomes for a measurement that can be performed on the system.
- Example: Specker's Triangle

$$(\{e, f, g\}, \{\{e, f\}, \{f, g\}, \{g, e\}\})$$



KS Contextuality in Test Spaces

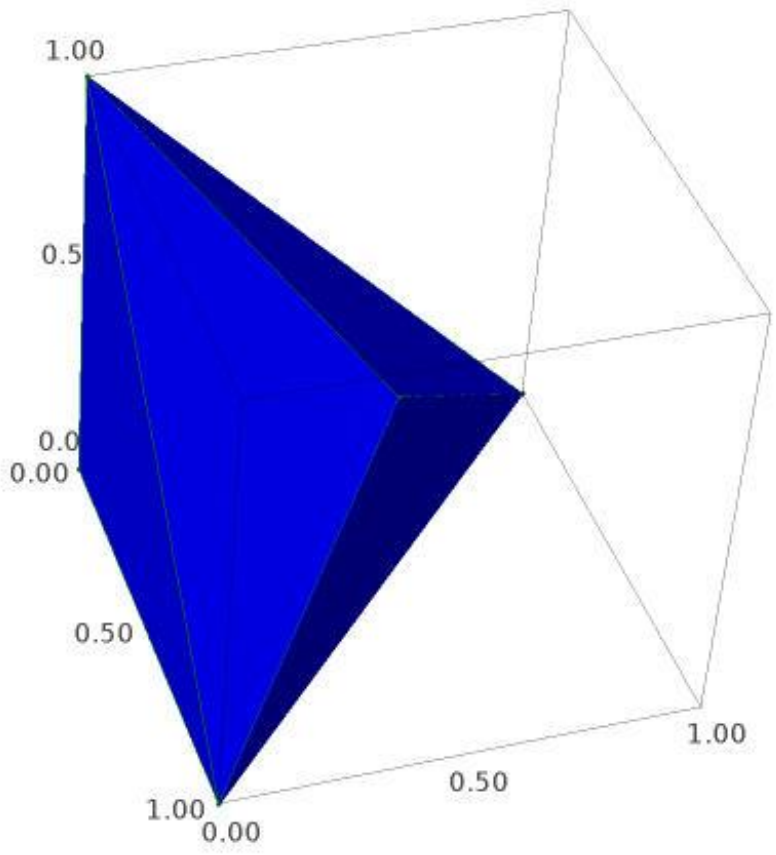
- A *state* on a test space is a function $\omega: X \to [0,1]$ such that
$$\forall E \in \Sigma, \qquad \sum_{e \in \mathrm{E}} \omega(e) = 1$$
- Let $\mathcal{S}(X,\Sigma)$ be the set of states on (X,E). For a finite test space this is a polytope.
- An *unnormalized state* on a test space is a function $\omega: X \to [0,1]$ such that
$$\forall E \in \Sigma, \qquad \sum_{e \in \mathrm{E}} \omega(e) \leq 1$$
- Let $\mathcal{S}_u(X,\Sigma)$ be the set of unnormalized states on (X,Σ). For a finite test space this is also a polytope.
- The advantage is that not all test spaces have states, but they do all have unnormalized states.
- Interpretation. We let our measurements sometimes fail, and not register an outcome. The probability of this happening can depend on which test we are measuring.

Example: Specker Triangle

- We proved previously that the only normalized state on a Specker triangle is
$$\omega(e) = \omega(f) = \omega(g) = \frac{1}{2}$$
- Unnormalized states just have to satisfy the inequalities

$$\omega(e) \geq 0, \qquad \omega(f) \geq 0, \qquad \omega(g) \geq 0$$

$$\omega(e) + \omega(f) \leq 1$$
$$\omega(f) + \omega(g) \leq 1$$
$$\omega(g) + \omega(e) \leq 1$$



Example Klyachko

- By a similar argument to Specker, the only normalized state is

$$\omega_j = \frac{1}{2} \text{ for } j = 0,1,2,3,4$$

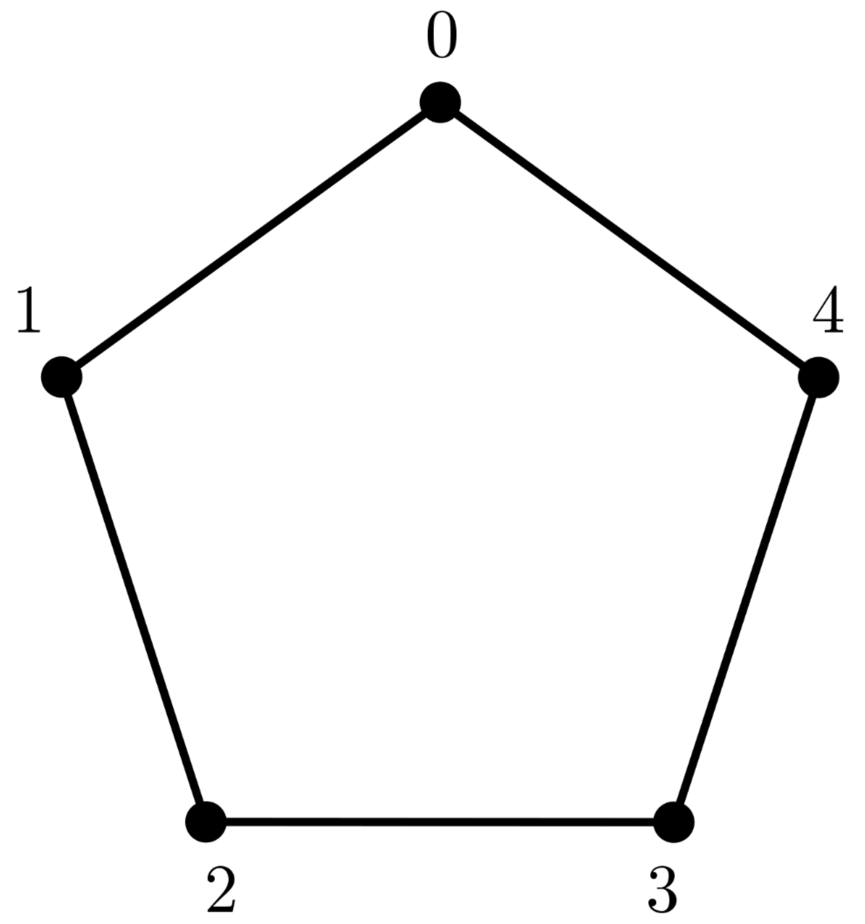
- For unnormalized states we have

$$\omega_j \geq 0$$
$$\omega_j + \omega_{j+1} \ (\mathrm{mod}\ 5) \leq 1$$

- From this, we can derive

$$\omega_0 + \omega_1 + \omega_2 + \omega_3 + \omega_4 \leq \frac{5}{2}$$

which is saturated by the normalized state.



KS Contextuality in Test Spaces

- A value function $v: X \to \{0,1\}$ on a test space is a function such that, for every test $E \in \Sigma$, $v(e) = 1$ for exactly one $e \in E$ and is 0 otherwise.
- A *KS noncontextual* state on (X, Σ) is a state ω that can be written as

$$\omega = \sum_j p_j v_j$$

where $p_j \geq 0$, $\sum_j p_j = 1$, and v_j is a value function.

- Let $\mathcal{C}(X, \Sigma)$ be the set of KS noncontextual states on (X, Σ).
- Clearly, $\mathcal{C}(X, \Sigma)$ is a polytope and $\mathcal{C}(X, \Sigma) \subseteq \mathcal{S}(X, \Sigma)$.
- The existence of KS contextuality proofs shows the inclusion is strict for some test spaces, e.g. 18 ray proof.

KS Contextuality in Test Spaces

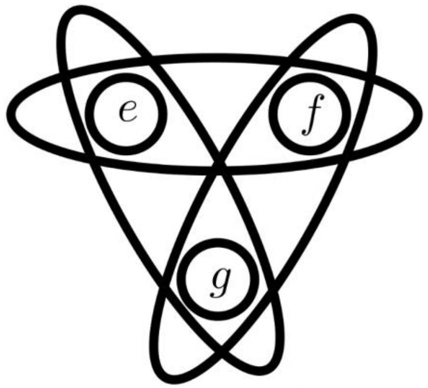
- An unnormalized value function $v: X \to \{0,1\}$ on a test space is a function such that, for every test $E \in \Sigma$, $v(e) = 1$ for at most one $e \in E$ and is 0 otherwise.
- A *unnormalized KS noncontextual* state on (X, Σ) is a state ω that can be written as

$$\omega = \sum_j p_j v_j$$

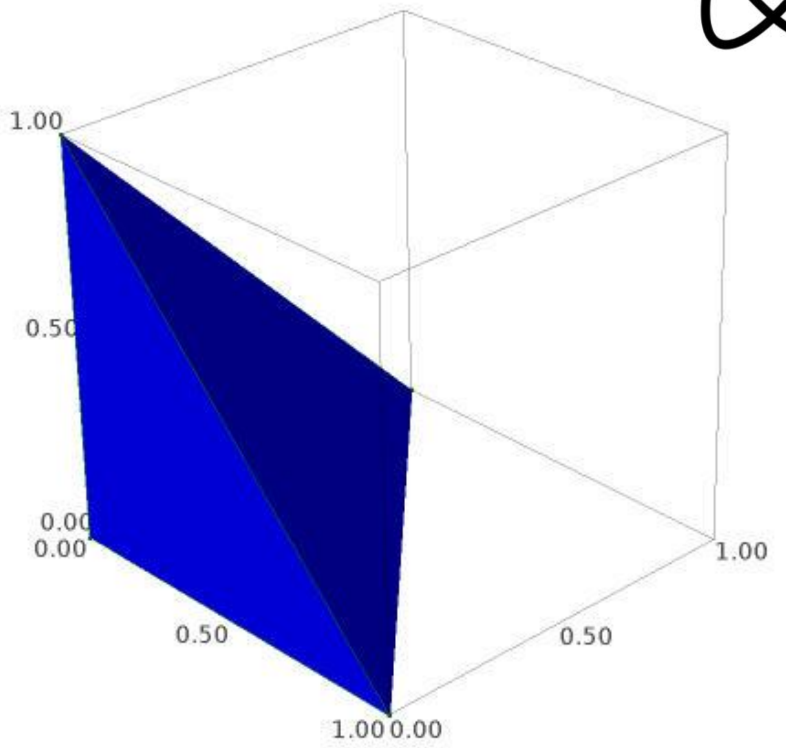
where $p_j \geq 0$, $\sum_j p_j = 1$, and v_j is an unnormalized value function.

- Let $\mathcal{C}_u(X, \Sigma)$ be the set of KS noncontextual states on (X, Σ).
- Clearly, $\mathcal{C}_u(X, \Sigma)$ is a polytope and $\mathcal{C}_u(X, \Sigma) \subseteq \mathcal{S}_u(X, \Sigma)$.
- The existence of Klyatchko style KS contextuality proofs shows the inclusion is strict for some test spaces.

Example: Specker Triangle



- There are no normalized states as if $v(e) = 1$ then
$$v(f) = v(g) = 0$$
but one of $v(f)$, $v(g)$ has to be 1.
- The unnormalized states are the polytope with extreme points
$$(0,0,0), (1,0,0), (0,1,0), (0,0,1)$$



- There are fewer KS noncontextual states than general states.

Example Klyachko

- There are no normalized KS noncontextual states.
- The unnormalized states form a polytope with extreme points

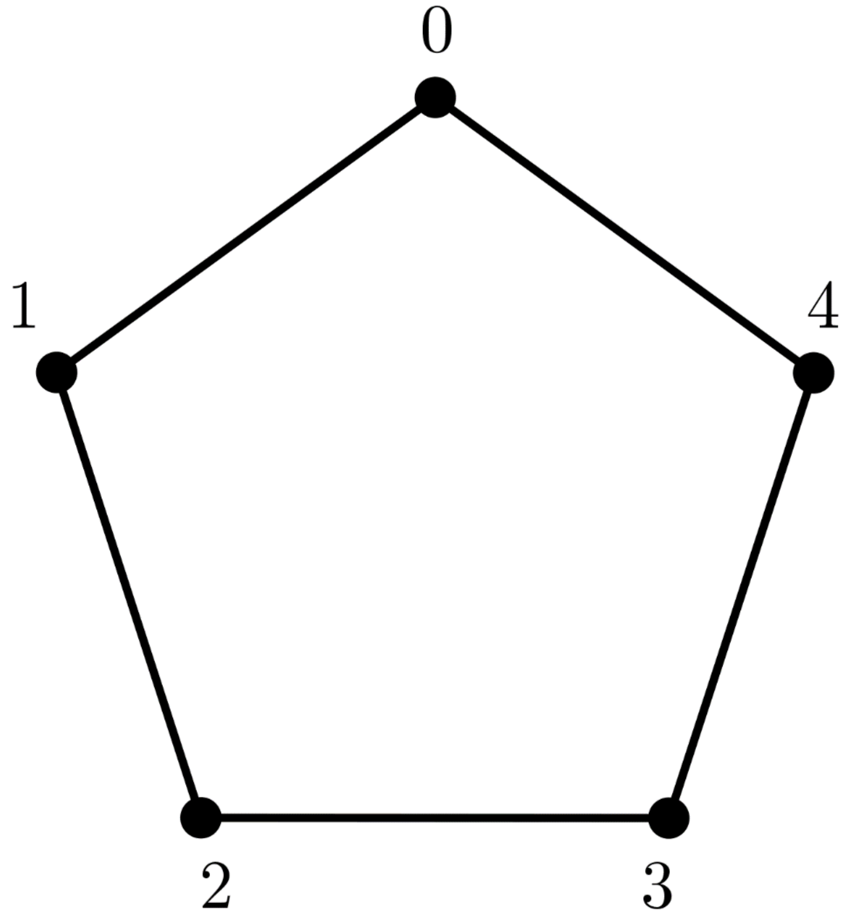
  $$(0,0,0,0,0)$$

  $(1,0,0,0,0)$ and cyclic permutations

  $(1,0,1,0,0)$ and cyclic permutations

  $(1,0,0,1,0)$ and cyclic permutations
- Unnnormalized KS noncontextual states satisfy

  $$\omega_0 + \omega_1 + \omega_2 + \omega_3 + \omega_4 \leq 2$$



KS Contextuality in Test Spaces

- A frame function f assigns a projector, $f(x) = \Pi_x$ to every outcome $x \in X$ such that, for every test $E \in \Sigma$,
  - If $x, y \in E$, $x \neq y$, then $\Pi_x \Pi_y = 0$
  - $\sum_{x \in \mathrm{E}} \Pi_x = I$
- A *quantum* state on (X, Σ) is a state ω that can be written as
$$\omega(x) = \mathrm{Tr}(\Pi_x \rho)$$
  for some frame function and some density matrix ρ.
- Let $\mathcal{Q}(X, \Sigma)$ be the set of quantum states on (X, Σ).
- $\mathcal{Q}(X, \Sigma)$ is a convex set, but not necessarily a polytope.
- $\mathcal{C}(X, \Sigma) \subseteq \mathcal{Q}(X, \Sigma) \subseteq \mathcal{S}(X, \Sigma)$ and the inclusions are strict for some test spaces.

$\mathcal{C}(X,\Sigma) \subseteq \mathcal{Q}(X,\Sigma)$

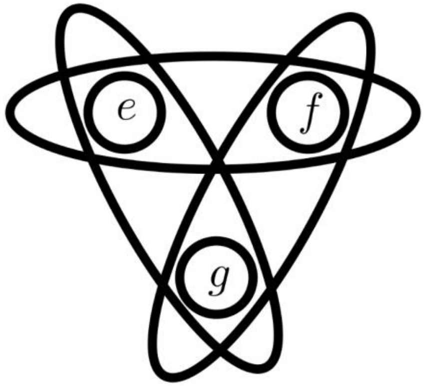
- Let $\mathcal{V}(X,\Sigma)$ be the set of value functions on (X,Σ).
- There are a finite number of them because X is finite.
- Let $(v_1, v_2, \cdots, v_N)$ be an ordering of the value functions.
- To $x \in X$, we assign the projector

$$\Pi_x = \begin{pmatrix} v_1(x) & 0 & \cdots & 0 \\ 0 & v_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & v_N(x) \end{pmatrix}$$

- For a classical state $\omega = \sum_j p_j v_j$, we assign the density matrix

$$\rho = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & p_N \end{pmatrix}$$

$\mathcal{Q}(X,\Sigma) \subseteq \mathcal{S}(X,\Sigma)$ is Strict for some Test Spaces



- Consider the Specker Triangle $(\{e,f,g\},\{\{e,f\},\{f,g\},\{g,e\}\})$
- We require projectors Π_e, Π_f, Π_g such that
$$\Pi_e\Pi_f = \Pi_e\Pi_g = \Pi_f\Pi_g = 0$$
- But then Π_e, Π_f, Π_g are mutually orthogonal, so
$$\Pi_e + \Pi_f + \Pi_g \leq I$$
- But this contradicts the requirement that
$$\Pi_e + \Pi_f = I$$

so there are no frame functions, and hence no quantum states, on the Specker triangle.

- Conclusion: There are theories that are more Kochen-Specker contextual than quantum theory.

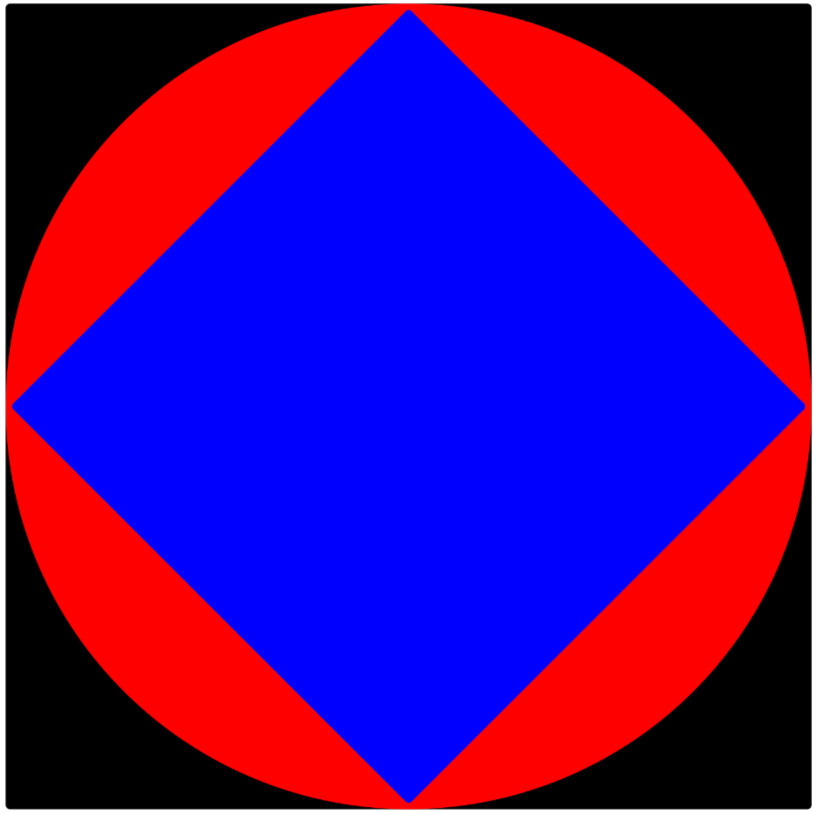
KS Contextuality in Test Spaces

- An unnormalized frame function f assigns a projector Π_x to every outcome $x \in X$ such that, for every test $E \in \Sigma$,
  - If $x, y \in E$, $x \neq y$, then $\Pi_x \Pi_y = 0$
  - $\sum_{x \in \mathrm{E}} \Pi_x \leq I$ (i.e. basis can be incomplete)
- An *unnormalized quantum* state on (X, Σ) is a state ω that can be written as

$$\omega(x) = \mathrm{Tr}(\Pi_x \rho)$$

  for some density matrix ρ.
- Let $\mathcal{Q}_u(X, \Sigma)$ be the set of quantum states on (X, Σ).
- $\mathcal{Q}_u(X, \Sigma)$ is a convex set, but not necessarily a polytope.
- $\mathcal{C}_u(X, \Sigma) \subseteq \mathcal{Q}_u(X, \Sigma) \subseteq \mathcal{S}_u(X, \Sigma)$ and the inclusions are strict for some test spaces.

The General Picture



$\mathcal{S}$

$\mathcal{Q}$

$\mathcal{C}$

- We can find inequalities satisfied by $\mathcal{C}$ or $\mathcal{C}_u$. These are noncontextuality inequalities.

  e.g. for Klyatchko $\sum_{j=0}^{4} \omega_j \leq 2$

- States in $\mathcal{Q}$ or $\mathcal{Q}_u$ may violate these inequalities.
- We can also find inequalities satisfied by $\mathcal{Q}$ or $\mathcal{Q}_u$.

$$\sum_{j=0}^{4} \omega_j \leq \sqrt{5}$$

- States in $\mathcal{S}$ or $\mathcal{S}_u$ may violate both sets of inequalities, but satisfy other inequalities

$$\sum_{j=0}^{4} \omega_j \leq \frac{5}{2}$$

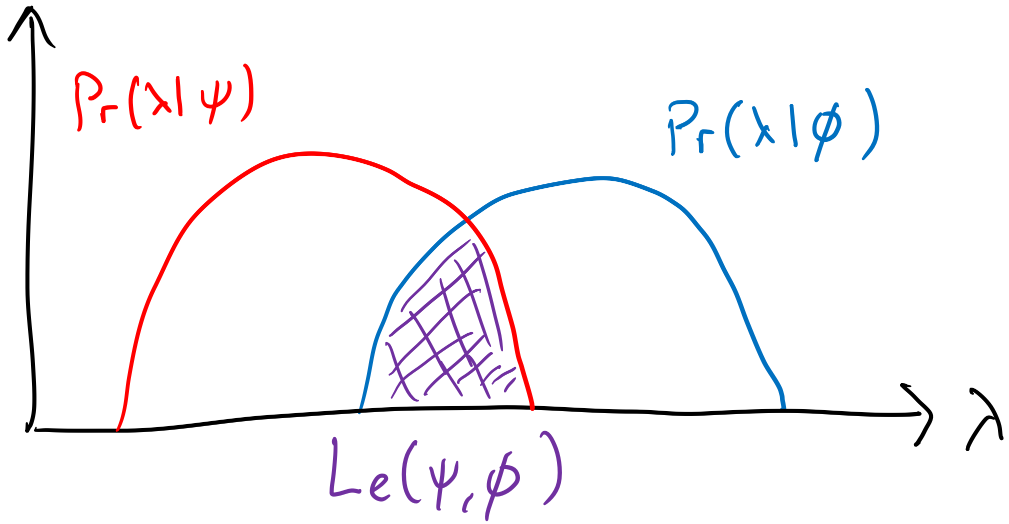
3.vi) Ψ-Ontology

- We now wish to investigate whether the (pure) quantum state has to be part of the ontology as it is in Beltrametti-Bugajski, the Bell model and de Broglie-Bohm theory.
- Our objective is to determine whether the kind of ψ-epistemic explanations that occur in the Spekkens toy theory can work in quantum theory.
- I will use naughty notation $\mathrm{Pr}(\lambda|\psi)$ for epistemic states:
  - We can only prove preparation contextuality for mixed states anyway.
  - What we will prove applies to *any* method of preparing $|\psi\rangle$, so it is best to avoid cluttering notation.

Definitions

- For two quantum states $|\psi\rangle$ and $|\phi\rangle$, we define their epistemic overlap in an ontological model as:

$$L_e(\psi, \phi) = \int_\Lambda d\lambda \; \min[\Pr(\lambda|\psi), \Pr(\lambda|\phi)]$$



Epistemic Overlap and Discrimination

- The optimal probability of correctly guessing whether $|\psi\rangle$ or $|\psi\rangle$ was prepared if you know λ is

$$p_{\text{succ}} = \frac{1}{2}(1 + D_c(\psi, \phi)) \quad \text{where} \quad D_c(\psi, \phi) = \frac{1}{2}\int_\Lambda |\Pr(\lambda|\psi) - \Pr(\lambda|\phi)|\, \mathrm{d}\lambda$$

- **Theorem**: $L_e(\psi, \phi) = 1 - D_c(\psi, \phi)$

- The operational interpretation of $L_e(\psi, \phi)$ is that, if you know λ, the optimal probability of correctly whether $|\psi\rangle$ or $|\psi\rangle$ was prepared if you know λ is

$$p_{\text{succ}} = \frac{1}{2}(2 - L_e(\psi, \phi))$$

Proof of Theorem

- If we define

$$\Lambda_{\psi>\phi} = \{\lambda|\Pr(\lambda|\psi) > \Pr(\lambda|\phi)\} \quad \text{and} \quad \Lambda_{\psi\leq\phi} = \{\lambda|\Pr(\lambda|\psi) \leq \Pr(\lambda|\phi)\}$$

then

$$D_c(\psi,\phi) = \frac{1}{2}\left(\Pr\left(\Lambda_{\psi>\phi}\middle|\psi\right) - \Pr\left(\Lambda_{\psi>\phi}\middle|\phi\right) + \Pr\left(\Lambda_{\psi\leq\phi}\middle|\phi\right) - \Pr\left(\Lambda_{\psi\leq\phi}\middle|\psi\right)\right)$$

$$= \frac{1}{2}\left(\left[1 - \Pr\left(\Lambda_{\psi\leq\phi}\middle|\psi\right)\right] - \Pr\left(\Lambda_{\psi>\phi}\middle|\phi\right) + \left[1 - \Pr\left(\Lambda_{\psi>\phi}\middle|\phi\right)\right] - \Pr\left(\Lambda_{\psi\leq\phi}\middle|\psi\right)\right)$$

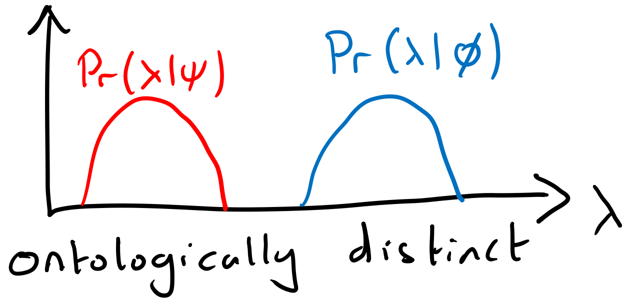
$$= 1 - \Pr\left(\Lambda_{\psi\leq\phi}\middle|\psi\right) - \Pr\left(\Lambda_{\psi>\phi}\middle|\phi\right)$$

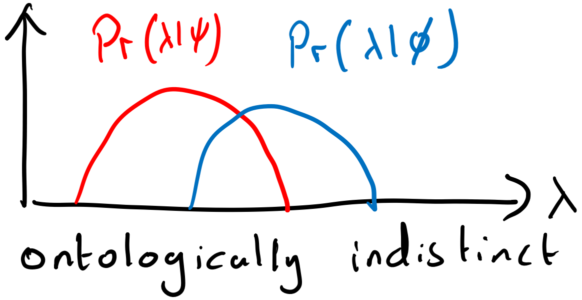
$$= 1 - \int_\Lambda d\lambda \ \min[\Pr(\lambda|\psi), \Pr(\lambda|\phi)]$$

$$= 1 - L_e(\psi,\phi)$$

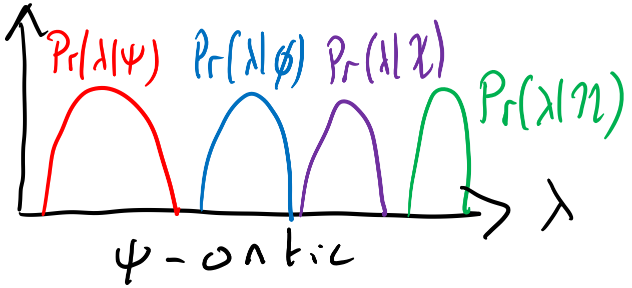
Definitions

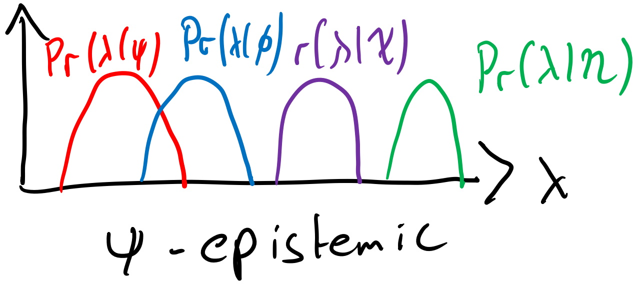
- $|\psi\rangle$ and $|\phi\rangle$ are ontologically distinct in an ontological model if $L_e(\phi, \psi) = 0$.





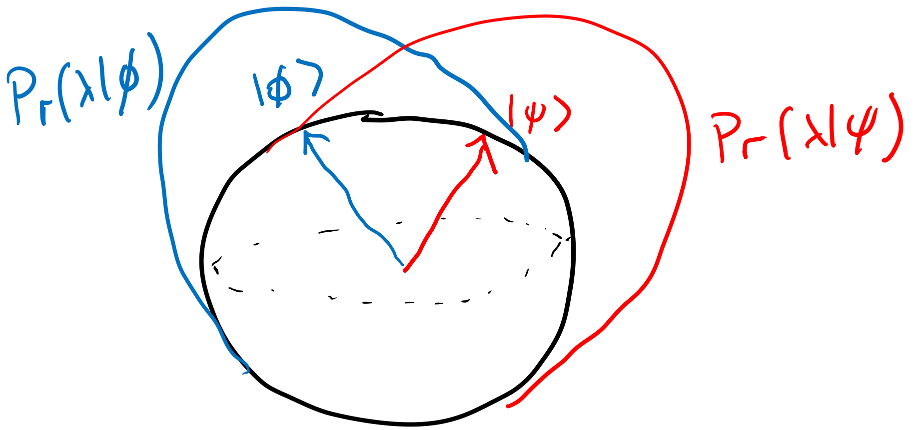
- An ontological model is called ψ-ontic if every pair of pure states in the model is ontologically distinct. Otherwise, it is called ψ-epistemic.





ψ-epistemic models exist

- ψ-epistemic models exist in all finite Hilbert space dimensions.
  - For d=2, the Kochen-Specker model is ψ-epistemic.



  - For d>2, it was proved by Lewis et. al. (Phys. Rev. Lett. 109:150404 (2012)) and Aaronson et. al. (*Phys. Rev. A* 88:032111 (2013)).

What next for ψ-ontology?

- Given that ψ-epistemic models exist, is that the end of the story? No.
  - We can try to prove something weaker than ψ-ontology, that still makes ψ-epistemic explanations implausible:

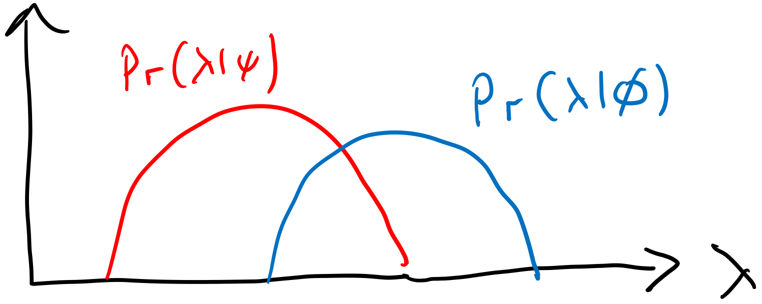
    $\Rightarrow$ non maximal ψ-epistemicity

  - We can add additional assumptions to the ontological models framework to prove ψ-ontology:

    $\Rightarrow$ Pusey-Barrett-Rudolph (PBR) theorem

Maximally ψ-epistemic models

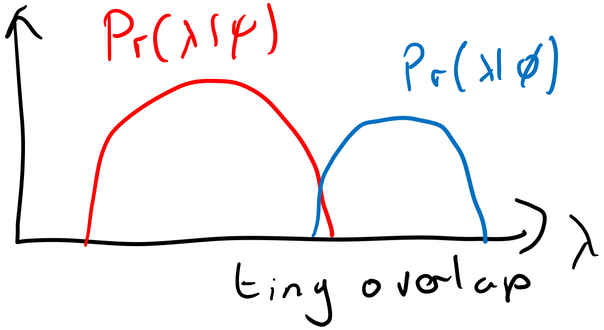
- Consider the ψ-epistemic explanation of the indistinguishability of quantum states:



$|\psi\rangle$ and $|\phi\rangle$ cannot be perfectly distinguished because sometimes the ontic state is exactly the same regardless of whether $|\psi\rangle$ or $|\phi\rangle$ was prepared.

- This explanation is rendered implausible if a suitable measure of the overlap of the probability distributions is small compared to a suitable measure of the overlap/indistinguishability of the quantum states.

i.e.



but $|\langle\phi|\psi\rangle|^2$ is large

$\Rightarrow$ this explanation plays almost no role.

Maximally ψ-epistemic models

- We need to be comparing measures of quantum and probability overlap that have a comparable operational meaning.
- We already have the epistemic overlap measure:

$$L_e(\psi,\phi) = \int_\Lambda d\lambda \ \min[\Pr(\lambda|\psi), \Pr(\lambda|\phi)]$$

- This measure has the following interpretation:
  - If the system is prepared in state $|\psi\rangle$ or state $|\phi\rangle$ with 50/50 probability and you don't know which, then if you knew the exact ontic state λ your optimal probability of guessing correctly is

$$p = \tfrac{1}{2}(2 - L_e(\psi,\phi))$$

- The comparable quantum overlap measure is:

$$L_q(\psi,\phi) = 1 - \sqrt{1 - |\langle\phi|\psi\rangle|^2}$$

  - If the system is prepared in state $|\psi\rangle$ or state $|\phi\rangle$ with 50/50 probability and you don't know which, then if you want to guess based on the outcome of a quantum measurement, your optimal probability of guessing correctly is

$$p = \tfrac{1}{2}(2 - L_q(\psi,\phi))$$

Maximally ψ-epistemic models

- An ontological model is maximally ψ-epistemic if, for every pair of pure states $|\psi\rangle$ and $|\phi\rangle$,

$$L_e(\psi,\phi) = L_q(\psi,\phi).$$

  - The indistinguishability of nonorthogonal states is entirely accounted for by the indistinguishability of the epistemic states.
- Spekkens' toy theory and the Kochen-Specker model are maximally ψ-epistemic.
- But such models can be ruled out for $d \geq 3$ using noncontextuality inequalities.

Ruling out Maximally ψ-epistemic models

- First note that $L_e(\psi,\phi) = \int_\Lambda d\lambda \min\{\Pr(\lambda|\psi), \Pr(\lambda|\phi)\} \leq \int_{\Lambda_\phi} d\lambda \Pr(\lambda|\psi)$

  where $\Lambda_\phi := \{\lambda \in \Lambda \,|\, \Pr(\lambda|\phi) > 0\}$

- We already showed that $\Lambda_\phi \subseteq \Gamma_\phi^M$ for any measurement M that has $|\phi\rangle$ as an outcome.

- Since this is true for all such M, we also have

$$\Lambda_\phi \subseteq \Gamma_\phi = \bigcap_{\{M \,|\, |\phi\rangle \in M\}} \Gamma_\phi^M$$

$$\therefore \; L_e(\psi,\phi) \leq \int_{\Gamma_\phi} d\lambda \Pr(\lambda|\psi) = \Pr(\Gamma_\phi|\psi)$$

Ruling out Maximally ψ-epistemic models

- Now if we consider a set of states $\{|\phi_j\rangle\}$ then we will have

$$\sum_j L_e(\psi,\phi_j) \leq \sum_j \Pr(\Gamma_{\phi_j}|\psi)$$ ← This is precisely what is bounded by a noncontextuality inequality

- We can then compute $\sum_j L_q(\psi,\phi_j)$ for the optimal states in the contextuality inequality. If $\sum_j \Pr(\Gamma_{\phi_j}|\psi) < \sum_j L_q(\psi,\phi_j)$ then maximally ψ-epistemic models are ruled out.
- It is better to compare the averages

$$\langle L_e\rangle = \frac{1}{n}\sum_{j=1}^{n} L_e(\psi,\phi_j) \qquad \langle L_q\rangle = \frac{1}{n}\sum_{j=1}^{n} L_q(\psi,\phi_j)$$

If $\langle L_q\rangle$ is large while $\langle L_e\rangle$ is small, the ψ-epistemic explanation of indistinguishability is in trouble.

Results from Various Contextuality Inequalities

	Dimension	No. states	$\langle L_e \rangle$	$\langle L_q \rangle$
Barrett et. al.	Prime power $d \geq 4$	d^2	$1/d^2$	$1-\sqrt{1-1/d}$
Leifer	$d \geq 3$	2^{d-1}	$1/2^{d-1}$	$1\sqrt{1-1/d}$
Branciard	$d \geq 4$	$n \geq 2$	$1/n$	$1-\sqrt{1-\frac{1}{4}n^{-1/(d-2)}}$
Amaral et. al.	$d \geq n_j$	$n_j \geq ?$	$n_j^{\delta-1}$	$1-\sqrt{\frac{1}{2}-\epsilon}$

J. Barrrett et. al., Phys. Rev. Lett. 112, 250403 (2014)
M. Leifer, Phys. Rev. Lett. 112, 160404 (2014)
C. Branciard, Phys. Rev. Lett. 113, 020409 (2014)
B. Amaral et. al., Phys. Rev. A 92, 062125 (2015)

Optimizing for $\langle L_q \rangle - \langle L_e \rangle$

	Optimal Dimension	Optimal No. states	Optimal $\langle L_q \rangle - \langle L_e \rangle$
Barrett et. al.	4	16	0.0715
Leifer	7	64	0.0586
Branciard	4	$n \to \infty$	0.134
Amaral et. al.	$d \to \infty$	$n_j \to \infty$	0.293

Is non maximal ψ-epistemicity significant?

- In any ontological model, there are two ways of explaining the indistinguishability of quantum states:
  - The epistemic states overlap.
  - Quantum measurements only reveal coarse-grained information about λ.
- It is not clear why the second explanation should not play some role in a ψ-epistemic theory.
- Therefore, I would say that we want to get $\langle L_q \rangle - \langle L_e \rangle$ as close to 1 as possible in order to convincingly rule out ψ-epistemic models.