Quantum Foundations Lecture 15

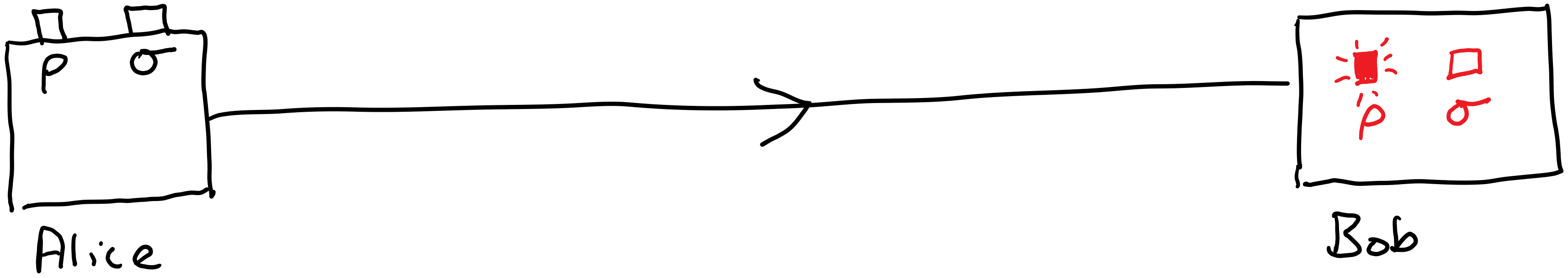
April 2, 2018

Dr. Matthew Leifer
leifer@chapman.edu
HSC112

Announcements

- Schmid College Academic Advising:
  - Tuesday April 3, 4:30pm-6:30pm AF209A (Prof. Leifer)
  - Wednesday April 4, 4:30pm-6:30pm Henley Hall Basement (Prof. Dressel)
- Adam Becker is returning to Chapman:
  - Book event and signing at 1888 center: Monday April 16. RSVP required https://bit.ly/AdamBecker
- Assignments
  - First Draft due on Blackboard April 11.
  - Peer review until April 16.
  - Discussion in class April 16.
  - Final Version due May 2.
- Homework 3 due April 11.
- I like lunch invitations

Application: Minimum Error Discrimination



- Alice has a preparation device that prepares the system in either the state ρ or the state σ. She chooses each with $50/50$ probability and sends the system to Bob.
- Bob makes a measurement on the system and has to guess whether ρ or σ was prepared.
- What is his maximum probability of success and what measurement should he make?

Classical Case

- Let's look at the classical case first. There is a variable that can take d possible values $j = 1,2,\cdots,d$.
- Alice prepares the probability distributions $\boldsymbol{p}$ or $\boldsymbol{q}$ with $50/50$ probability.
- Bob sees the value of j and has to guess whether $\boldsymbol{p}$ or $\boldsymbol{q}$ was prepared.
- Bob decides on a subset $E_{\boldsymbol{p}} \subseteq \{1,2,\cdots,d\}$. If $j \in E_{\boldsymbol{p}}$ he guesses $\boldsymbol{p}$. If it is in the complement $E_{\boldsymbol{q}} = \{1,2,\cdots,d\} \backslash E_{\boldsymbol{p}}$, he guesses $\boldsymbol{q}$.

Classical Case

- Therefore,

$$p_{\text{succ}} \leq \frac{1}{2}\left[1 + \sum_{\{j|p_j \geq q_j\}} (p_j - q_j)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2}\left(\sum_{\{j|p_j \geq q_j\}} (p_j - q_j) + \sum_{\{j|p_j < q_j\}} (q_j - p_j)\right)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2}\sum_{j=1}^{d} |p_j - q_j|\right]$$

$$= \frac{1}{2}[1 + D_c(\boldsymbol{p}, \boldsymbol{q})]$$

- Where $D_c(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2}\sum_{j=1}^{d}|p_j - q_j|$ is called the *variational distance*.

Quantum Case

- **Theorem** (Helstrom, Holevo): The optimal success probability in the quantum case is given by

$$p_{\mathrm{succ}} = \frac{1}{2}\left[1 + D_q(\rho, \sigma)\right]$$

where

$$D_q(\rho, \sigma) = \frac{1}{2}\mathrm{Tr}(|\rho - \sigma|)$$

is known as the *trace distance*, and the matrix norm is given by

$$|M| = \sqrt{M^\dagger M}.$$

Quantum Case

- **Lemma**: If M is a Hermitian operator then
$$\mathrm{Tr}(|M|) = \sum_j |\lambda_j|$$
where λ_j are the eigenvalues of M.
- Proof: Let $M = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ be the spectral decomposition of M and write
$$M = \sum_{\{j|\lambda_j \geq 0\}} |\lambda_j||\phi_j\rangle\langle\phi_j| - \sum_{\{k|\lambda_k < 0\}} |\lambda_k||\phi_k\rangle\langle\phi_k|$$
Then,
$$|M| = \sqrt{M^\dagger M} = \sqrt{M^2} \text{ because } M \text{ is Hermitian.}$$

Quantum Case

$$|M| = \sqrt{M^2}$$

$$= \sqrt{\left(\sum_{\{j|\lambda_j \geq 0\}} |\lambda_j||\phi_j\rangle\langle\phi_j| - \sum_{\{k|\lambda_k<0\}} |\lambda_k||\phi_k\rangle\langle\phi_k|\right)\left(\sum_{\{l|\lambda_l \geq 0\}} |\lambda_l||\phi_l\rangle\langle\phi_l| - \sum_{\{m|\lambda_m<0\}} |\lambda_m||\phi_m\rangle\langle\phi_m|\right)}$$

$$= \sqrt{\sum_{\{j|\lambda_j \geq 0\}} \sum_{\{l|\lambda_l \geq 0\}} |\lambda_j||\lambda_l||\phi_j\rangle\langle\phi_j|\phi_l\rangle\langle\phi_l| + \sum_{\{k|\lambda_k<0\}} \sum_{\{m|\lambda_m<0\}} |\lambda_k||\lambda_m||\phi_k\rangle\langle\phi_k|\phi_m\rangle\langle\phi_m|}$$

- Note: cross terms are zero because the subspaces with $\lambda_j \geq 0$ and $\lambda_j < 0$ are orthogonal.

$$|M| = \sqrt{\sum_{\{j|\lambda_j \geq 0\}} \sum_{\{l|\lambda_l \geq 0\}} |\lambda_j||\lambda_l|\delta_{jl}|\phi_j\rangle\langle\phi_l| + \sum_{\{k|\lambda_k<0\}} \sum_{\{m|\lambda_m<0\}} |\lambda_k||\lambda_m|\delta_{km}|\phi_k\rangle\langle\phi_m|}$$

$$= \sqrt{\sum_{\{j|\lambda_j \geq 0\}} |\lambda_j|^2|\phi_j\rangle\langle\phi_j| + \sum_{\{k|\lambda_k<0\}} |\lambda_k|^2|\phi_k\rangle\langle\phi_k|}$$

$$= \sqrt{\sum_j |\lambda_j|^2|\phi_j\rangle\langle\phi_j|} = \sum_j |\lambda_j||\phi_j\rangle\langle\phi_j|$$

- And hence $\mathrm{Tr}(|M|) = \sum_j |\lambda_j|$.

Quantum Case

- Now consider Bob's strategy. He has to choose a two-outcome POVM: $E_\rho, E_\sigma = I - E_\rho$, such that if he gets the outcome E_ρ he will guess ρ and if he gets the outcome E_σ he will guess σ.
- His success probability is

$$\begin{aligned} p_{\mathrm{succ}} &= \mathrm{Prob}(\rho \text{ is prepared})\mathrm{Prob}(E_\rho|\rho) + \mathrm{Prob}(\sigma \text{ is prepared})\mathrm{Prob}(E_\sigma|\sigma) \\ &= \frac{1}{2}\left[\mathrm{Tr}(E_\rho\rho) + \mathrm{Tr}(E_\sigma\sigma)\right] \\ &= \frac{1}{2}\left[\mathrm{Tr}(E_\rho\rho) + \mathrm{Tr}\left((I - E_\rho)\sigma\right)\right] \\ &= \frac{1}{2}\left[\mathrm{Tr}(\sigma) + \mathrm{Tr}\left(E_\rho(\rho - \sigma)\right)\right] \\ &= \frac{1}{2}\left[1 + \mathrm{Tr}\left(E_\rho(\rho - \sigma)\right)\right] \end{aligned}$$

Quantum Case

- Now let $\rho - \sigma = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ be the spectral decomposition of $\rho - \sigma$.

$$\mathrm{Tr}\big(E_\rho(\rho - \sigma)\big) = \mathrm{Tr}\left(E_\rho\left(\sum_j \lambda_j |\phi_j\rangle\langle\phi_j|\right)\right)$$

$$= \sum_j \lambda_j \mathrm{Tr}\big(E_\rho|\phi_j\rangle\langle\phi_j|\big) = \sum_j \lambda_j \langle\phi_j|E_\rho|\phi_j\rangle$$

- Now, $0 \leq \langle\phi_j|E_\rho|\phi_j\rangle \leq 1$, so this is clearly maximized if we can choose $\langle\phi_j|E_\rho|\phi_j\rangle = 1$ for $\lambda_j \geq 0$ and $\langle\phi_j|E_\rho|\phi_j\rangle = 0$ for $\lambda_j < 0$. This can be achieved if we choose

$$E_\rho = P_+ = \sum_{\{j|\lambda_j \geq 0\}} |\phi_j\rangle\langle\phi_j|$$

Quantum Case

- So we have
$$\begin{aligned}\mathrm{Tr}\big(E_\rho(\rho-\sigma)\big) &\le \mathrm{Tr}(P_+(\rho-\sigma))\\ &= \sum_j \lambda_j \langle \phi_j | P_+ | \phi_j \rangle\\ &= \sum_j \sum_{\{k|\lambda_k \ge 0\}} \lambda_j \langle \phi_j | \phi_k \rangle \langle \phi_k | \phi_j \rangle = \sum_{\{j|\lambda_j \ge 0\}} |\lambda_j|\end{aligned}$$
- However, ρ and σ are both density matrices, so
$$\mathrm{Tr}(\rho-\sigma) = \mathrm{Tr}(\rho) - \mathrm{Tr}(\sigma) = 1 - 1 = 0$$
- Therefore $\sum_j \lambda_j = \sum_{\{j|\lambda_j \ge 0\}} |\lambda_j| - \sum_{\{j|\lambda_j < 0\}} |\lambda_j| = 0$ or
$$\sum_{\{j|\lambda_j \ge 0\}} |\lambda_j| = \sum_{\{j|\lambda_j < 0\}} |\lambda_j|$$

Quantum Case

- Hence,

$$\sum_{\{j|\lambda_j\ge 0\}} |\lambda_j| = \frac{1}{2}\left(\sum_{\{j|\lambda_j\ge 0\}} |\lambda_j| + \sum_{\{j|\lambda_j< 0\}} |\lambda_j|\right)$$
$$= \frac{1}{2}\sum_j |\lambda_j|$$

- Now, if we apply the lemma, this gives

$$\sum_{\{j|\lambda_j\ge 0\}} |\lambda_j| = \frac{1}{2}\mathrm{Tr}(|\rho-\sigma|) = D_q(\rho,\sigma)$$

- Putting it all together gives

$$p_{\text{succ}} \le \frac{1}{2}\big(1 + D_q(\rho,\sigma)\big)$$

with equality achieved if Bob chooses $E_\rho = P_+$, i.e. the projector onto the positive eigenspace of $\rho - \sigma$.

Special Cases

- Note that if ρ and σ are diagonal in the same basis

$$\rho = \sum_j p_j |j\rangle\langle j| \quad \text{and} \quad \sigma = \sum_j q_j |j\rangle\langle j|$$

  then the eigenvalues of $\rho - \sigma$ are $p_j - q_j$ and we get

$$D_q(\rho, \sigma) = \frac{1}{2} \sum_j \left| p_j - q_j \right| = D_c(\boldsymbol{p}, \boldsymbol{q})$$

  recovering the classical result.

- If $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ are both pure states then (you will prove on Hwk. 4)

$$D_q(\rho, \sigma) = \sqrt{1 - |\langle\phi|\psi\rangle|^2}$$

- Therefore, pure states are perfectly distinguishable iff $\langle\phi|\psi\rangle = 0$.

8.vi) The Lindblad Equation

- A density operator evolves under unitary dynamics according to

$$\rho \longrightarrow U \rho U^\dagger$$

- If the unitary is generated by a fixed Hamiltonian $U(t) = e^{-iH(t-t_0)}$ then

$$\rho(t) = e^{-iH(t-t_0)} \rho(t_0) e^{iH(t-t_0)}$$

$$\rho(t+\Delta t) - \rho(t) = [I - iH\Delta t]\rho(t)[I + iH\Delta t] - \rho(t) \quad \text{to 1st order}$$

$$= -i\Delta t\left(H\rho(t) - \rho(t)H\right)$$

$$= -i\Delta t\,[H, \rho(t)]$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} = -i[H, \rho]}$$

This is called the von-Neuman equation.

Continuous Time Dynamics

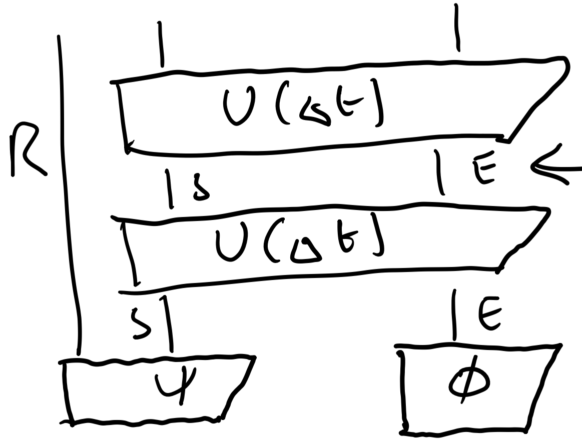
- But we know that finite time dynamics need not be unitary. We can have a completely positive, trace preserving map.

$$\rho \to \mathcal{E}(\rho) = \sum_j M^{(j)} \rho M^{(j)\dagger}$$

- What is the corresponding continuous-time dynamics?
- You might have thought that we can just parameterize $\mathcal{E}$ by t and assume that $\mathcal{E}_{t+\Delta t} = \mathcal{E}_{\Delta t} \circ \mathcal{E}_t$ i.e. $\rho(t_0+t+\Delta t) = \mathcal{E}_{\Delta t}\big(\mathcal{E}_t(\rho(t_0))\big)$
- This would give $\mathcal{E}_t$ the structure of a continuous semi-group.
- But there is a problem with this from the point of view of the larger church.

The View from the Larger Church

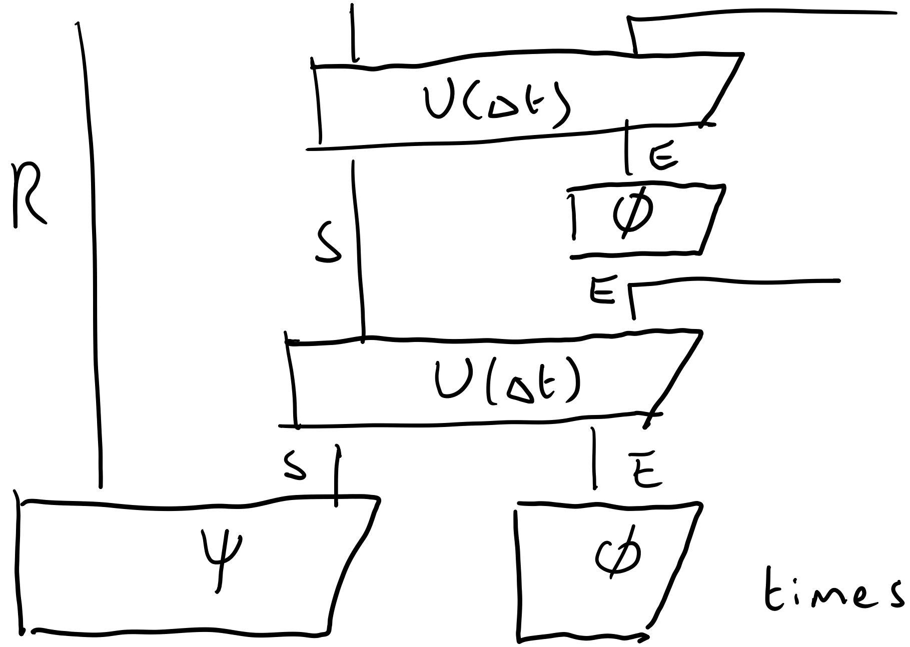
- Recall that, in order to derive CPT maps, we assumed that the system was initially uncorrelated from its environment.
- Thus, if we want $\mathcal{E}_{2\Delta t} = \mathcal{E}_{\Delta t} \circ \mathcal{E}_{\Delta t}$ with $\mathcal{E}_{\Delta t}$ CPT, we need the system to be uncorrelated with its environment after every Δt timestep.
- If the system is interacting with the environment under a fixed Hamiltonian H_{SE} then this won't be true in general



Expect a correlated state at this stage already.

The View from the Larger Church

- So we will have to assume that the interaction with the environment is approximately like this



- The system behaves as if it is interacting with a new uncorrelated environment at every time step.
- This is called the weak coupling limit.
- E.g. suppose the environment is a thermal bath

timescale for rethermalization of the bath $\ll$ timescale on which system gets significantly correlated with environment.

Deriving the Lindblad Equation

- $\mathcal{E}_{\Delta t}$ will have the usual operator sum form

$$\rho(t+\Delta t) = \mathcal{E}_{\Delta t}(\rho(t)) = \sum_{j=0}^{N} M^{(j)} \rho(t) M^{(j)\dagger} \simeq \rho(t) + O(\Delta t)$$

- We want to expand each term up to order Δt.
- We can, without loss of generality, put all of the $O(1)$ term in a single Kraus operator

$$M^{(0)} = I + \underbrace{(L^{(0)} - iH)}\Delta t + O(\Delta t^2)$$

general decomposition of an operator into two Hermitian operators

- In order for $M^{(j)} \rho M^{(j)\dagger}$ to contribute for $j = 1, 2, \ldots, N$ we need

$$M^{(j)} = L^{(j)} \sqrt{\Delta t} + O(\Delta t)$$

Deriving the Lindblad Equation

- Plugging these terms into $\rho(t+\Delta t) = \mathcal{E}_{\Delta t}(\rho(t))$ gives

$$\rho(t+\Delta t) - \rho(t) = \left[(L^{(0)} - iH)\rho(t) + \rho(t)(L^{(0)} + iH) + \sum_{j=1}^{N} L^{(j)} \rho(t) L^{(j)\dagger} \right] \Delta t$$

$$= \Big[\underbrace{-i[H,\rho]}_{\text{unitary part}} + \underbrace{\{L^{(0)}, \rho(t)\}}_{\text{anticommutator}} + \sum_{j=1}^{N} L^{(j)} \rho(t) L^{(j)\dagger} \Big] \Delta t$$

$L^{(0)}\rho(t) + \rho(t) L^{(0)}$

$$\therefore \quad \frac{d\rho}{dt} = -i[H,\rho] + \{L^{(0)}, \rho(t)\} + \sum_{j=1}^{N} L^{(j)} \rho(t) L^{(j)\dagger}$$

Deriving the Lindblad Equation

- We still have to impose the trace preserving condition

$$\sum_j M^{(j)\dagger} M^{(j)} = I$$

$$M^{(0)\dagger} M^{(0)} = \left[I + (L^{(0)} + iH)\Delta t\right]\left[I + (L^{(0)} - iH)\Delta t\right]$$

$$= I + (L^{(0)} + \cancel{iH} + L^{(0)} - \cancel{iH})\Delta t + O(\Delta t^2)$$

$$= I + 2L^{(0)}\Delta t + O(\Delta t^2)$$

$$\sum_{j=1}^{N} M^{(j)\dagger} M^{(j)} = \left(\sum_{j=1}^{N} L^{(j)\dagger} L^{(j)}\right)\Delta t + O(\Delta t^2)$$

$$\Rightarrow L^{(0)} = -\frac{1}{2}\sum_{j=1}^{N} L^{(j)\dagger} L^{(j)}$$

$$\boxed{\frac{d\rho}{dt} = -i[H, \rho] + \sum_{j=1}^{N}\left(L^{(j)}\rho L^{(j)\dagger} - \frac{1}{2}\left\{L^{(j)\dagger}L^{(j)}, \rho\right\}\right)}$$

Example: Decoherence

- Consider a qubit with Hamiltonian $H=0$ and a single Lindblad operator
  $$L = \gamma \sigma_3$$
- Then we get $\frac{d\rho}{dt} = \gamma^2 (\sigma_3 \rho \sigma_3 - \rho)$
- In terms of components $\begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix} = \begin{pmatrix} 0 & -2\gamma^2 \rho_{01} \\ -2\gamma^2 \rho_{10} & 0 \end{pmatrix}$

so we get the solution;

$$\rho(t) = \begin{pmatrix} \rho_{00}(0) & \rho_{01}(0) e^{-2\gamma^2 t} \\ \rho_{10}(0) e^{-2\gamma^2 t} & \rho_{11}(0) \end{pmatrix}$$

The off-diagonal elements decay exponentially

System decoheres in the $|0\rangle, |1\rangle$ basis.

9) Ontological Models

- The aim of this section is to investigate the possibility of constructing a realist theory (known as an *ontological model*) that can reproduce the predictions of quantum theory.
- We start with a simple toy-model that reproduces many of the apparently puzzling phenomena we have studied so far: The Spekkens' toy theory.
- These phenomena are naturally explained if there is a restriction on the amount of information we can have about the ontic state (an "epistemic restriction" or "epistriction") and the quantum state is epistemic.
- After this we will present the general definition of an ontological model and prove a number of no-go theorems that imply that a realist theory underlying quantum theory cannot be like this.

9) Ontological Models

- Good references for this section include:
  - David Jennings and Matthew Leifer, “No Return to Classical Reality”, Contemporary Physics, vol. 57, iss. 1, pp. 60-82 (2015) https://doi.org/10.1080/00107514.2015.1063233 preprint: https://arxiv.org/abs/1501.03202
  - Robert W. Spekkens, “Quasi-Quantization: Classical Statistical Theories with an Epistemic Restriction”, in “Quantum Theory: Informational Foundations and Foils”, Giulio Chiribella and Robert W. Spekkens (eds.), pp. 83-135, Springer (2015) preprint: https://arxiv.org/abs/1409.5041
  - Robert W. Spekkens, “Contextuality for preparations, transformations, and unsharp measurements”, Physical Review A, vol. 71 052108 (2005). Preprint: https://arxiv.org/abs/quant-ph/0406166
  - J. S. Bell, “Speakable and Unspeakable in Quantum Mechanics”, 2nd edition, Cambridge University Press (2004).
  - Matthew Leifer, “Is the Quantum State Real? An Extended Review of ψ-ontology Theorems”, Quanta, vol. 3, no. 1, pp. 67-155 (2014). http://dx.doi.org/10.12743/quanta.v3i1.22

9) Ontological Models

9. Ontological Models
    i. Epistricted Theories
    ii. Definitions
    iii. Examples
    iv. Excess Baggage
    v. Contextuality
    vi. Ψ-ontology
    vii. Bell's Theorem
    viii. The Colbeck-Renner Theorem