Quantum Foundations Lecture 14

March 28, 2018

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HSC112

Announcements

- Schmid College Academic Advising:
  - Tuesday April 3, 4:30pm-6:30pm AF209A (Prof. Leifer)
  - Wednesday April 4, 4:30pm-6:30pm Henley Hall Basement (Prof. Dressel)
- Adam Becker is returning to Chapman:
  - Book event and signing at 1888 center: Monday April 16. RSVP required https://bit.ly/AdamBecker
- Assignments
  - First Draft due on Blackboard April 11.
  - Peer review until April 16.
  - Discussion in class April 16.
  - Final Version due May 2.
- Homework 3 due April 11.
- I like lunch invitations

Review of Last Lecture

The View from the Smaller Church

The View from the Smaller Church

The View from the Smaller Space

The View from the Smaller Church

The View from the Smaller Church

The View from the Smaller Church

The View from the Smaller Church

Trace Preservation

Trace Preservation

Positivity vs. Complete Positivity

Positivity vs. Complete Positivity

Complete Positivity

Complete Positivity

Complete Positivity

Summary

Examples of Qubit CPT Maps

8.v) Postivie Operator Valued Measures (POVMs)

The View from the Larger Church

The View from the Larger Church

The View from the Smaller Church

The View from the Smaller Church

Summary

Example

Application: Minimum Error Discrimination

- Alice has a preparation device that prepares the system in either the state ρ or the state σ. She chooses each with $50/50$ probability and sends the system to Bob.
- Bob makes a measurement on the system and has to guess whether ρ or σ was prepared.
- What is his maximum probability of success and what measurement should he make?

Classical Case

- Let's look at the classical case first. There is a variable that can take d possible values $j = 1,2,\cdots,d$.
- Alice prepares the probability distributions $\boldsymbol{p}$ or $\boldsymbol{q}$ with $50/50$ probability.
- Bob sees the value of j and has to guess whether $\boldsymbol{p}$ or $\boldsymbol{q}$ was prepared.
- Bob decides on a subset $E_{\boldsymbol{p}} \subseteq \{1,2,\cdots,d\}$. If $j \in E_{\boldsymbol{p}}$ he guesses $\boldsymbol{p}$. If it is in the complement $E_{\boldsymbol{q}} = \{1,2,\cdots,d\} \backslash E_{\boldsymbol{p}}$, he guesses $\boldsymbol{q}$.

Classical Case

- What is his probability of success:
  $p_{\text{succ}} = \text{Prob}(\boldsymbol{p} \text{ is prepared})\text{Prob}(E_{\boldsymbol{p}}|\boldsymbol{p}) + \text{Prob}(\boldsymbol{q} \text{ is prepared})\text{Prob}(E_{\boldsymbol{q}}|\boldsymbol{q})$
  $= \frac{1}{2}\left[\sum_{j\in E_{\boldsymbol{p}}} p_j + \sum_{j\in E_{\boldsymbol{q}}} q_j\right]$
- However, $\sum_{j\in E_{\boldsymbol{q}}} q_j = 1 - \sum_{k\in E_{\boldsymbol{p}}} q_j$ and so

$$p_{\text{succ}} = \frac{1}{2}\left[1 + \sum_{j\in E_{\boldsymbol{p}}} (p_j - q_j)\right]$$

- Clearly, p_{succ} is largest if we choose $E_{\boldsymbol{p}}$ to contain all and only those j's such that $p_j \geq q_j$

Classical Case

- Therefore,

$$p_{\text{succ}} \le \frac{1}{2}\left[1 + \sum_{\{j|p_j \ge q_j\}} (p_j - q_j)\right]$$

which can be achieved if Bob guesses $\boldsymbol{p}$ whenever $p_j \ge q_j$ and $\boldsymbol{q}$ otherwise.

- We can rewrite this in a simpler way by noting that

$$\sum_{\{j|p_j \ge q_j\}} p_j + \sum_{\{j|p_j < q_j\}} p_j = 1 \quad \text{and} \quad \sum_{\{j|p_j \ge q_j\}} q_j + \sum_{\{j|p_j < q_j\}} q_j = 1$$

- Subtracting these and rearranging gives

$$\sum_{\{j|p_j \ge q_j\}} (p_j - q_j) = \sum_{\{j|p_j < q_j\}} (q_j - p_j)$$

Classical Case

- Therefore,

$$p_{\text{succ}} \le \frac{1}{2}\left[1 + \sum_{\{j|p_j \ge q_j\}} (p_j - q_j)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2}\left(\sum_{\{j|p_j \ge q_j\}} (p_j - q_j) + \sum_{\{j|p_j < q_j\}} (q_j - p_j)\right)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2}\sum_{j=1}^{d} \left|p_j - q_j\right|\right]$$

$$= \frac{1}{2}\left[1 + D_c(\boldsymbol{p}, \boldsymbol{q})\right]$$

- Where $D_c(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2}\sum_{j=1}^{d}\left|p_j - q_j\right|$ is called the *variational distance*.

Quantum Case

- **Theorem** (Helstrom, Holevo): The optimal success probability in the quantum case is given by

$$p_{\text{succ}} = \frac{1}{2}\left[1 + D_q(\rho,\sigma)\right]$$

where

$$D_q(\rho,\sigma) = \frac{1}{2}\mathrm{Tr}(|\rho - \sigma|)$$

is known as the *trace distance*, and the matrix norm is given by

$$|M| = \sqrt{M^\dagger M}.$$