Quantum Foundations Lecture 13

March 26, 2018

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HSC112

Announcements

- Adam Becker is returning to Chapman:
  - Book event and signing at 1888 center: Monday April 16. RSVP required https://bit.ly/AdamBecker
- Assignments
  - First Draft due on Blackboard April 11.
  - Peer review until April 16.
  - Discussion in class April 16.
  - Final Version due May 2.
- Homework 3 due April 11.
- I like lunch invitations

8) The Generalized Formalism

AKA Everything I taught you in PHYS451 is wrong

i. The Two Churches of Quantum Theory
ii. The Hilbert Space of Hermitian Matrices
iii. Density Operators
iv. Completely Positive Maps
v. Positive Operator Valued Measures
vi. Quantum Instruments
vii. The Lindblad Equation

The Generalized Formalism

- In undergraduate quantum mechanics, we normally assume:
  - The system does not interact with its environment unless it is being measured.
  - Measurements are of the most ideal kind possible.
  - We have perfect knowledge of what our experimental devices are doing.
- These assumptions are never true in practice. When they do not hold, we have to generalize the formalism.
- We have already seen part of this in the GPT section: density matrices and POVMs. We will review them again, but there is much more.
- Supplementary reading for this section:

  - Teiko Heinosaari and Mario Ziman, "The Mathematical Language of Quantum Theory", Cambridge University Press (2012)
  - Benjamin Schumacher and Michael Westmoreland, "Quantum Processes, Systems, and Information", Cambridge University Press (2010)
  - Michael Nielsen and Isaac Chuang, "Quantum Computation and Quantum Information", Cambridge University Press (2000)

3.i) The Two Churches of Quantum Theory

- The Church of The Larger Hilbert Space:
  - Quantum theory is a dynamical theory, akin to a classical field theory, but with a weirder object called the wavefunction in place of a classical field.
  - All is to be derived from a quantum state (of the universe in principle) evolving unitarily according to the Schrödinger equation.
  - Today, we will allow projective measurements as well, but see lecture on Everett/many-worlds for how to derive them.
- The Church of The Smaller Hilbert Space:
  - Something strange has happened to our physical variables: they have become noncommutative.
  - Quantum theory is the only consistent probability theory for such variables.
- In this section, we will give both churches views on each construction.

3.ii) The Hilbert Space of Hermitian Matrices

- As it is a Hilbert space $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ must have multiple orthonormal bases.
- The standard basis that we have been using is just $|j,k\rangle_{AB} = |j\rangle_B \otimes {}_A\langle k|$

$$\langle jk|lm\rangle_{AB} = \left(|j\rangle_B \otimes {}_A\langle k|\right)^\dagger \left(|l\rangle_B \otimes {}_A\langle m|\right) = \langle j|l\rangle_B \langle m|k\rangle_A = \delta_j^l \, \delta_m^k$$

- Clearly, $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ has dimension $d_A \times d_B$
- But there are other bases, e.g. consider $\mathcal{L}(\mathcal{H}_A)$ with $\mathcal{H}_A = \mathbb{C}^2$ and let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $S_j = \frac{1}{\sqrt{2}} \sigma_j$ is an orthonormal basis as $\mathrm{Tr}(S_j^\dagger S_h) = \delta_{jh}$

- Consequently, every 2×2 operator can be written as $M = \frac{1}{2} \sum_j m_j \sigma_j$ with $m_j = \mathrm{Tr}(\sigma_j M)$

The Space of Hermitian Matrices

- If input and output spaces are the same, we can have Hermitian matrices

$$M^\dagger = M \quad \text{with} \quad M = \sum_{jk} M_{jk} |j\rangle_A \otimes {}_A\langle k| \qquad M^\dagger = \sum_{jk} M_{jk}^* |k\rangle_A \otimes {}_A\langle j|$$

- The set of Hermitian matrices on $\mathcal{H}_A$, denoted $\mathcal{S}(\mathcal{H}_A)$ is a Hilbert space over $\mathbb{R}$.

i.e. if $M^\dagger = M$ $N^\dagger = N$ then $(\alpha M + \beta N)^\dagger = \alpha M + \beta N$

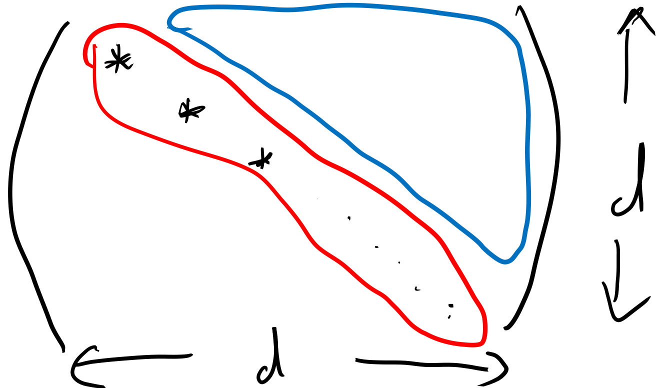
so long as $\alpha, \beta \in \mathbb{R}$

and $\mathrm{Tr}(M^\dagger N) \in \mathbb{R}$

- The dimension of this space is d^2

d real parameters $+$ $(d-1)d$ real parameters

$= d^2$ real parameters



Hermitian Bases

- The matrices $|j\rangle\langle k|$ are not Hermitian, but there must be a basis of d^2 Hermitian matrices.
- We already saw $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  which is a Hermitian basis for operators on $\mathbb{C}^2$
- In general, you can take

$$\left.\begin{array}{l} \frac{1}{\sqrt{2}}\left(|j\rangle\langle k| + |k\rangle\langle j|\right) \\ \frac{1}{\sqrt{2}}\left(i|j\rangle\langle k| - i|k\rangle\langle j|\right) \end{array}\right\}$$

These are not all different and not all nonzero (e.g. take $j=k$)

Proper counting gives d^2 orthonormal matrices.

Hermitian Matrices are Self-Dual

- Because $\mathcal{S}(\mathcal{H}_A) \subseteq \mathcal{L}(\mathcal{H}_A) \cong \mathcal{H}_A \otimes \mathcal{H}_A^\dagger$, the dual $\mathcal{S}(\mathcal{H}_A)^\dagger$ won't be all of $\mathcal{L}(\mathcal{H}_A)$
- Because $\mathcal{S}(\mathcal{H}_A)$ is a <u>real</u> Hilbert space, $\mathcal{S}(\mathcal{H}_A)^\dagger$ consists of linear functionals from $\mathcal{S}(\mathcal{H}_A)$ to $\mathbb{R}$, not $\mathbb{C}$
- Because $\mathcal{S}(\mathcal{H}_A)$ is a Hilbert space, the inner product still induces an isomorphism $\mathcal{S}(\mathcal{H}_A) \cong \mathcal{S}(\mathcal{H}_A)^\dagger$

Write $M \in \mathcal{S}(\mathcal{H}_A)$ in terms of a self-adjoint basis

$$M = \sum_j m_j N_j$$

real coefficients

Then $M^\dagger = \sum_j m_j N_j^\dagger = \sum_j m_j N_j$ so a vector in $\mathcal{S}(\mathcal{H}_A)$ is its own dual vector

The Space of Commutative Matrices

- Consider a maximal set of commuting matrices on $\mathcal{H}_A$, i.e. the set of operators that are diagonal in a common basis. This is a Hilbert space (over $\mathbb{C}$)
- If $|j\rangle$ is the diagonalizing basis then $|j\rangle\langle j|$ is a matrix basis, since all matrices can be written as

$$M = \sum_j \underbrace{\lambda_j}_{\text{eigenvalues}} |j\rangle\langle j|$$

dimension is d.

- If we restrict attention to Hermitian commuting operators (λ_j's real) then this is also a Hilbert space, now over $\mathbb{R}$. Denote this space as $\mathcal{C}(\mathcal{H}_A)$. (Matrices in $\mathcal{C}(\mathcal{H}_A)$ are also their own duals)

3.iii) Density Operators

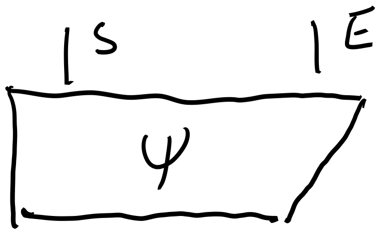
- According to the larger church, the universe always has a pure state vector $|\psi\rangle$.
- If any other mathematical object is used for a quantum state, it must be because we are looking at a subsystem.
- State space is $\mathcal{H}_S \otimes \mathcal{H}_E$

system we are interested in

Environment

and we have $|\psi\rangle_{SE} = \sum_{jk} \psi^{jk} |j\rangle_S \otimes |k\rangle_E$ or $\psi^{j_S k_E}$

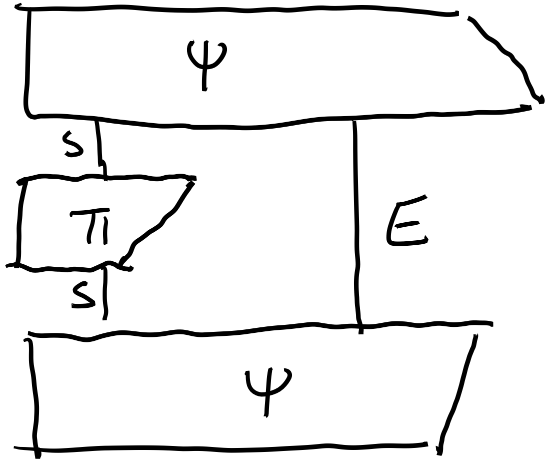
or

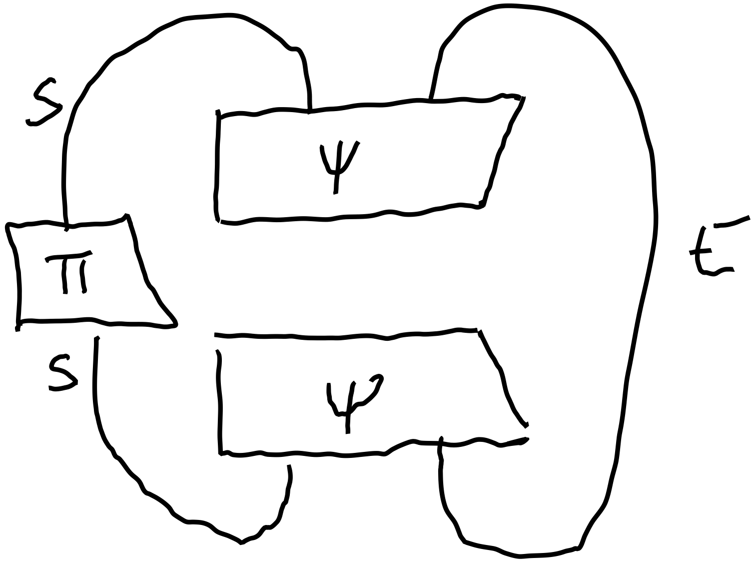


The View from the Larger Church

1. Suppose we make a projective measurement on system S alone. The probability of getting outcome corresponding to projector Π is

$${}_{SE}\langle\psi|\,\Pi_S\otimes I_E\,|\psi\rangle_{SE} \;=\; \psi^{\dagger}_{j_S k_E}\,\Pi^{j_S}_{l_S}\,\psi^{l_S k_E} \;=$$





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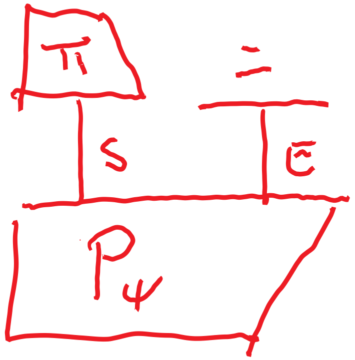
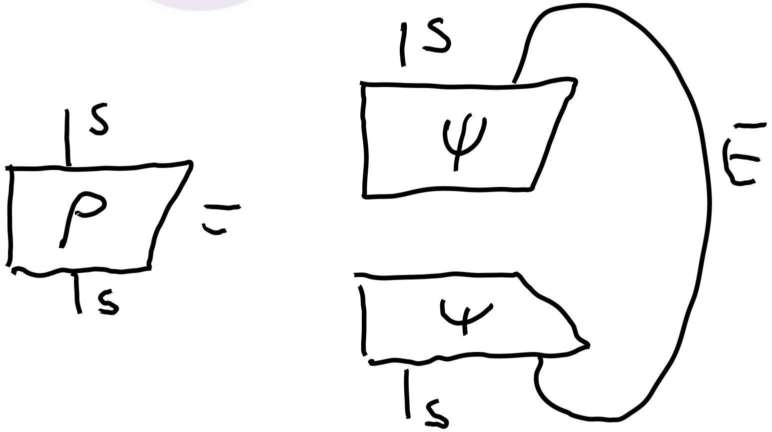


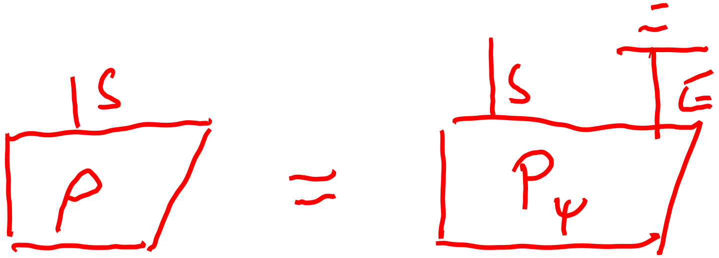
where $P_\psi = |\psi\rangle\langle\psi|$

The View From The Larger Church

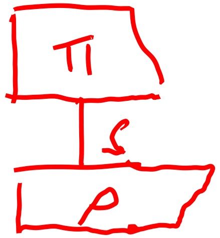
- If we define the object

$$\rho_S = \mathrm{Tr}_E(|\psi\rangle\langle\psi|) \qquad \rho^{i_S}_{j_S} = \psi^{i_S k_E}\,\psi^{\dagger}_{j_S k_E}$$





then the probability is $\mathrm{Prob}(\Pi) = \mathrm{Tr}(\Pi_S \rho_S) = \Pi^{k_S}_{j_S}\,\rho^{j_S}_{k_S} =$



- ρ lives in the space $\mathcal{H}_S \otimes \mathcal{H}_S^{\dagger} \equiv \mathcal{L}(\mathcal{H}_S) \equiv \mathcal{L}(\mathcal{H}_S)^{\dagger}$

  so it is both an operator and a duperator

- We normally call it a <u>density operator</u> (although we use it as a duperator)

An aside on positive operators

- A positive operator $M \in \mathcal{L}(\mathcal{H}_A)$ is an operator that satisfies

$$ {}_A\langle \psi | M | \psi \rangle_A \geq 0 \quad \text{for all} \quad |\psi\rangle_A \in \mathcal{H} $$

- Theorem: An operator is positive iff it is self-adjoint and has positive (≥ 0) eigenvalues

Proof:

Let $|\psi\rangle = |\phi\rangle + i|\chi\rangle$

$$ \Rightarrow \langle \phi | M | \phi \rangle - i \langle \chi | M | \phi \rangle + i \langle \phi | M | \chi \rangle + \langle \chi | M | \chi \rangle \geq 0 \quad (1) $$

Let $|\psi\rangle = |\phi\rangle - i|\chi\rangle$

$$ \langle \phi | M | \phi \rangle + i \langle \chi | M | \phi \rangle - i \langle \phi | M | \chi \rangle + \langle \chi | M | \chi \rangle \geq 0 \quad (2) $$

$\frac{(2) - (1)}{2i}$: $\langle \chi | M | \phi \rangle = \langle \phi | M | \chi \rangle$ which is the definition of self-adjoint.

An aside on positive operators

A self adjoint operator has real eigenvalues

Let $M|\phi\rangle = \lambda|\phi\rangle$

By positivity $\langle\phi|M|\phi\rangle \geq 0 \Leftrightarrow \lambda \geq 0$

Conversely, if $M = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ with $\lambda_j \geq 0$ then

$$\langle\psi|M|\psi\rangle = \sum_j \lambda_j \langle\psi|\phi_j\rangle\langle\phi_j|\psi\rangle = \sum_j \lambda_j |\langle\phi_j|\psi\rangle|^2 \geq 0.$$

⊙ Theorem: An operator $M \in \mathcal{L}(\mathcal{H}_A)$ is positive iff it can be written as

$$M = N^\dagger N \quad \text{where} \quad N \in \mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$$

Proof: If $M = N^\dagger N$ then $\langle\psi|M|\psi\rangle = \langle\psi|N^\dagger N|\psi\rangle = \| N|\psi\rangle\|^2 \geq 0$

Conversely, if $M = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ then let $M^{1/2} = \sum_j \sqrt{\lambda_j}\, |\psi_j\rangle\langle\psi_j|$

and then $M = N^\dagger N$ for $N = M^{1/2}$

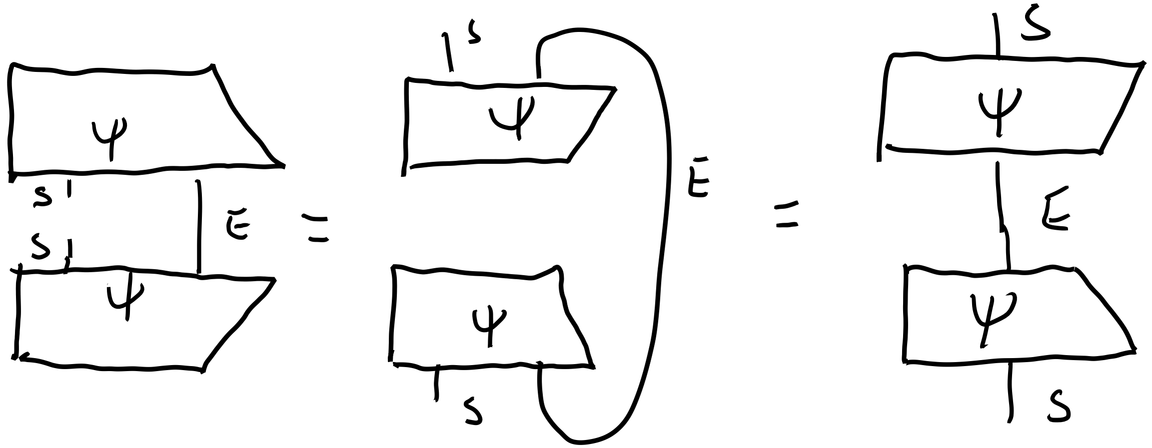
Properties of the density operator

- Given that $\psi^{j_S h_E}$ is normalized $\quad \psi^{j_S h_E}\,\psi^{\dagger}_{j_S h_E} = 1$

$$\therefore \mathrm{Tr}(\rho) = \rho^{j_S}_{j_S} = \psi^{j_S h_E}\,\psi^{\dagger}_{j_S h_E} = 1$$

- We can also write $\quad \rho^{k_S}_{j_S} = \psi^{k_S}_{l_E}\,\psi^{\dagger\, l_E}_{j_S}$

Proof: $\psi^{k_S l_E}\,\psi^{\dagger}_{j_S l_E} = \psi^{k_S}_{m_E}\,\delta^{m_E l_E}\,\delta_{n_E l_E}\,\psi^{\dagger\, n_E}_{j_S} = \psi^{k_S}_{m_E}\,\delta^{m_E}_{n_E}\,\psi^{\dagger\, n_E}_{j_S} = \psi^{k_S}_{n_E}\,\psi^{\dagger\, n_E}_{j_S}$



(Don't have a great diagrammatic notation for a partial transpose)

Properties of the density operator

- $\rho^{k_s}_{j_s} = \psi^{\dagger\, l_E}_{j_s}\, \psi^{k_s}_{l_E} = \psi^{k_s}_{l_E}\, \psi^{\dagger\, l_E}_{j_s}$ is of the form $N^\dagger N$ with $N = \psi^{\dagger\, l_E}_{j_s} \in \mathcal{L}(\mathcal{H}_S \to \mathcal{H}_E)$

  so $\rho^{k_s}_{j_s}$ is a positive operator.

- In summary, density operators must be positive and have Trace $= 1$.

- Can any positive, trace 1 operator arise from ignoring the environment for some $|\psi\rangle_{SE}$.

  Yes. Define $|\psi_\rho\rangle_{SS'} = \rho^{1/2}_S |\delta\rangle_{SS'}$ or $\psi_\rho{}^{j_s k_s} = (\rho^{1/2})^{j_s}_{l_s}\, \delta^{l_s}_{k_s}$

  This is an example of a purification of a density operator.

The View from the Smaller Church

- According to the smaller church, a quantum state should be <u>any</u> consistent way of assigning probabilities to observables.
- We can view a quantum state as a functional that assigns expectation values to observables

$$\rho : S(\mathcal{H}_A) \longrightarrow \mathbb{R}$$

- When we apply it to projection operators, we should get probabilities.
- Classically, expectation values behave linearly

$$\langle \alpha X + \beta Y \rangle = \alpha \langle X \rangle + \beta \langle Y \rangle$$

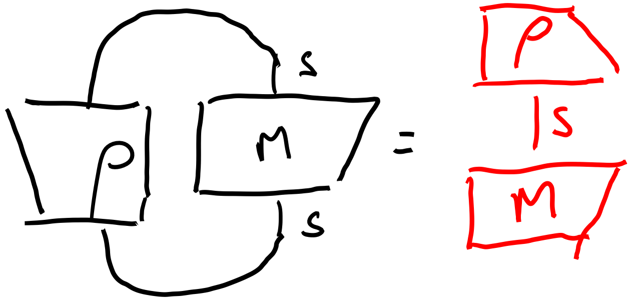
- We will impose this for quantum observables too (but can remove this later)

$$\rho(\alpha M + \beta N) = \alpha \rho(M) + \beta \rho(N)$$

The View from the Smaller Church

- A linear functional from $\tilde{S}(\mathcal{H}_A)$ to $\mathbb{R}$ is the definition of $\tilde{S}(\mathcal{H}_A)^{\dagger}$, so ρ must be a duperator.
- However, we already saw that $\tilde{S}(\mathcal{H}_A)$ is self-dual, so we get for free that ρ is a self-adjoint operator

$$\rho(M) = \rho^{j_s}_{k_s} M^{k_s}_{j_s} = \mathrm{Tr}(\rho M) =$$



- Since projectors must get assigned probabilities

$$\mathrm{Tr}(\rho \Pi) \geq 0 \qquad \rho^{j_s}_{k_s} \Pi^{k_s}_{j_s} \geq 0$$

- Let Π be a 1-dimensional projector $\quad \Pi^{k_s}_{j_s} = \psi^{\dagger}_{j_s} \psi^{k_s}$

Then $\quad \psi^{\dagger}_{j_s} \rho^{j_s}_{k_s} \psi^{k_s} = \langle \psi | \rho | \psi \rangle \geq 0 \quad$ which is positivity.

The View from the Smaller Church

- Finally a projective measurement $\{\Pi_k\}$ $\sum_k \Pi_k = I$ must receive probabilities that sum to 1

$$1 = \sum_k \mathrm{Tr}(\rho \Pi_k) = \mathrm{Tr}\left(\rho \left[\sum_k \Pi_k\right]\right) = \mathrm{Tr}(\rho I) = \mathrm{Tr}(\rho)$$

so ρ must have trace $= 1$.

- Note: If we apply the same reasoning to $\mathcal{C}(\mathcal{H}_A)$ instead of $\mathcal{S}(\mathcal{H}_A)$, we would get commuting density operators, all of the form

$$\rho = \sum_j p_j |j\rangle\langle j| \leftarrow \text{diagonal in same basis}$$

$\uparrow$ Classical probabilities

$\mathcal{C}(\mathcal{H}_A) \longrightarrow \mathcal{S}(\mathcal{H}_A)$

Classical probability $\longrightarrow$ Quantum theory.

Removing the Linearity Condition

- The linearity condition $\rho(M+N) = \rho(M) + \rho(N)$ is not operationally meaningful when M and N do not commute

  We can't measure $M+N$ by measuring M at the same time as N and then adding the results.

- Fortunately it can be removed.

- Gleason's Theorem (which is hard to prove) states that:

  For Hilbert space dimension ≥ 3 any function from projectors to $\mathbb{R}$ that satisfies

  $$f(\Pi) \geq 0 \quad , \quad f(\Pi_1 + \Pi_2) = f(\Pi_1) + f(\Pi_2) \text{ if } \Pi_1 \Pi_2 = 0$$

  $$f(I) = 1$$

  is of the form $f(\Pi) = \mathrm{Tr}(\rho \Pi)$ for some density operator ρ.

Qubit Density Operators

- We have already seen that any qubit operator can be written as

$$M = \frac{1}{2}\sum_j m_j \sigma_j = \frac{1}{2}\left(m_0 I + m_1\sigma_1 + m_2\sigma_2 + m_3\sigma_3\right)$$

- Density operators must have +ve eigenvalues and $Tr(\rho) = 1$

$$\Longleftrightarrow \quad \rho = \frac{1}{2}\left(I + \sum_{j=1}^{3} r_j \sigma_j\right) \quad \text{with} \quad \sum_j |r_j|^2 \leq 1 \quad (=1 \text{ for a pure state})$$

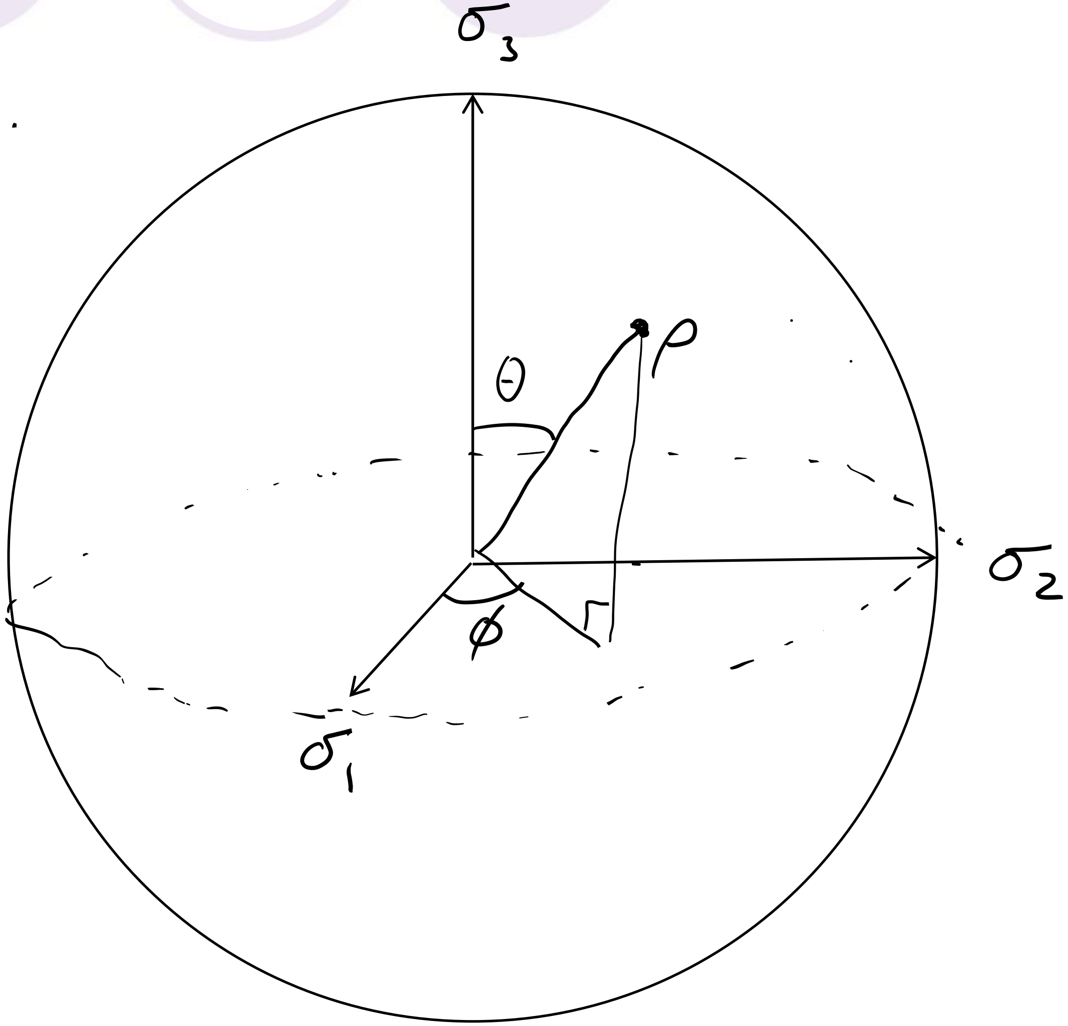
coefficient of 1 comes from $Tr(\rho) = 1$

Comes from +ve eigenvalues

- Density operators are points inside the unit ball (pure states are on the surface)

Qubit Density Operators

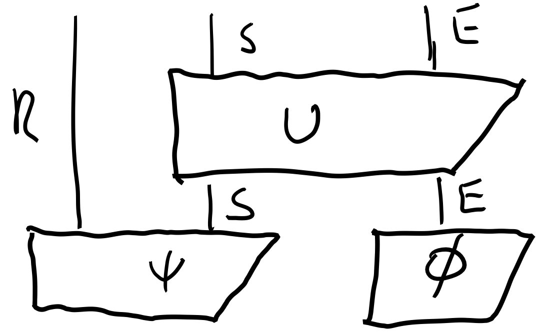
- This gives the Bloch sphere representation.
- Pure states are on the surface
- Mixed states are inside



Note: The geometry is much more complicated in higher dimensions

8.iv) Completely Positive Maps

- The dynamics of an isolated system is unitary, but in general a system might interact with its environment. How do we keep track of the state (density operator) of the system on its own?
- According to the larger church, the system and environment generally start in a (possibly entangled) pure state $|\psi\rangle_{SR} \in \mathcal{H}_S \otimes \mathcal{H}_R$
- However, the description to be given here only works if the system interacts with a part of the environment it is initially uncorrelated with, so we assume $|\psi\rangle_{SR} \otimes |\phi\rangle_E \in \mathcal{H}_S \otimes \mathcal{H}_R \otimes \mathcal{H}_E$ and the dynamics is $|\tilde{\psi}\rangle_{SRE} = U_{SE} |\psi\rangle_{SR} \otimes |\phi\rangle_E$



where $|\tilde{\psi}\rangle_{SRE}$ is the final state of SRE.

The View from the Larger Church

- We are only interested in keeping track of the density operator of S.

Initially: $\rho_S = \mathrm{Tr}_R\left(|\psi\rangle_{SR}\langle\psi|\right)$

After U: $\tilde{\rho}_S = \mathrm{Tr}_{RE}\left(|\tilde{\psi}\rangle_{SRE}\langle\tilde{\psi}|\right)$

$$= \mathrm{Tr}_{RE}\left(U_{SE}|\psi\rangle_{SR}|\phi\rangle_E \; {}_{SR}\langle\psi|\,{}_E\langle\phi|U^\dagger_{SE}\right)$$

$$= \mathrm{Tr}_E\left(U_{SE}\,\rho_S \otimes |\phi\rangle_E\langle\phi|\,U^\dagger_{SE}\right)$$

$$= \sum_j {}_E\langle j|U_{SE}|\phi\rangle_E \;\rho_S\; {}_E\langle\phi|U^\dagger_{SE}|j\rangle_E$$

$$= \sum_j M^{(j)}\rho_S M^{(j)\dagger} \leftarrow$$ This is called the operator sum decomposition

where $M^{(j)} = {}_E\langle j|U_{SE}|\phi\rangle_E$ are called Kraus operators

The View from the Larger Church

- The Kraus operators have to satisfy

$$\sum_j M^{(j)\dagger} M^{(j)} = \sum_j {}_E\langle\phi| U_{SE}^\dagger |j\rangle_E \langle j| U_{SE} |\phi\rangle_E = {}_E\langle\phi| U_{SE}^\dagger U_{SE} |\phi\rangle_E$$

$$= {}_E\langle\phi| I_{SE} |\phi\rangle_E = {}_E\langle\phi|\phi\rangle_E \, I_S = I_S$$

- Do they have to satisfy any other constraints?

No. For any set of operators $M^{(j)} \in \mathcal{L}(\mathcal{H}_S)$ s.t. $\sum_j M^{(j)\dagger} M^{(j)} = I_S$

you can construct a unitary U_{SE}

$$M^{(j)} = {}_E\langle j| U_{SE} |\phi\rangle_E$$

(see e.g. Nielsen and Chuang for proof)

The View from the Smaller Church

- According to the smaller church, dynamics should be any mapping of states to states that leads to well-defined probabilities for all observables at the output.
- This turns out to be remarkably subtle.
- Firstly, we will allow the output Hilbert space $\mathcal{H}_B$ to be different from the input Hilbert space $\mathcal{H}_A$

  We may add a new subsystem or discard part of the system during the dynamics.
- So we need some sort of map $\mathcal{E}_{B|A}$ from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$ that maps density operators to density operators.

The View from the Smaller Church

- We will demand that $\mathcal{E}_{B|A}$ is linear. Why?

  If we prepare ρ_A with probability p
  or σ_A with probability $(1-p)$

  Then $\mathcal{E}_{B|A}(p\rho_A + (1-p)\sigma_A) = p\,\mathcal{E}_{B|A}(\rho_A) + (1-p)\,\mathcal{E}_{B|A}(\sigma_A)$
- Strictly speaking, this only means that $\mathcal{E}_{B|A}$ has to be affine, i.e. acts linearly on positive linear combinations.
- But you can always extend an affine map to a linear one just by defining $\mathcal{E}_{B|A}(-\rho_A) = -\mathcal{E}_{B|A}(\rho_A)$
- So, we will have a linear operator from linear operators to linear operators

  $\mathcal{E}_{B|A} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B))$ Sometimes called a superoperator.

The View from the Smaller Space

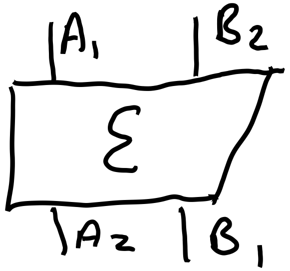
- Now comes the fun part:

$$\mathcal{L}(\mathcal{H}_A) = \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1}^\dagger \qquad \mathcal{L}(\mathcal{H}_B) = \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_1}^\dagger$$

These are both $\mathcal{H}_A$ but it helps to keep track of which is the input and which is the output

$$\therefore \mathcal{L}(\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)) = \mathcal{L}(\mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1}^\dagger \to \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_1}^\dagger) = \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_1}^\dagger \otimes \mathcal{H}_{A_2}^\dagger \otimes \mathcal{H}_{A_1}$$

$$\therefore \mathcal{E}_{B|A} = \sum_{jklm} \mathcal{E}^{jm}_{kl}\, |j\rangle_{A_1} \otimes \langle k|_{A_2} \otimes \langle l|_{B_1} \otimes |m\rangle_{B_2} \qquad \mathcal{E}^{j_{A_1} m_{B_2}}_{k_{A_2} l_{B_1}}$$



$$\mathcal{E}_{B|A} = \sum_{(jk)(lm)} \mathcal{E}^{(jk)}_{(lm)}\, |jk\rangle_A \otimes \langle lm|_B$$