Quantum Foundations Lecture 12

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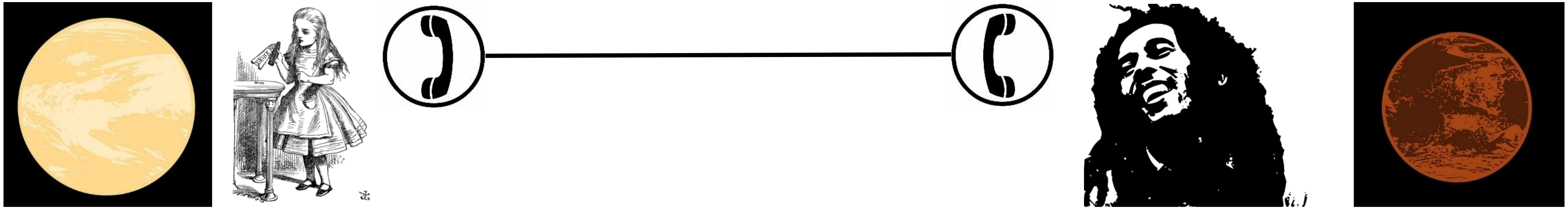
HSC112

7.x) Application: Quantum Teleportation



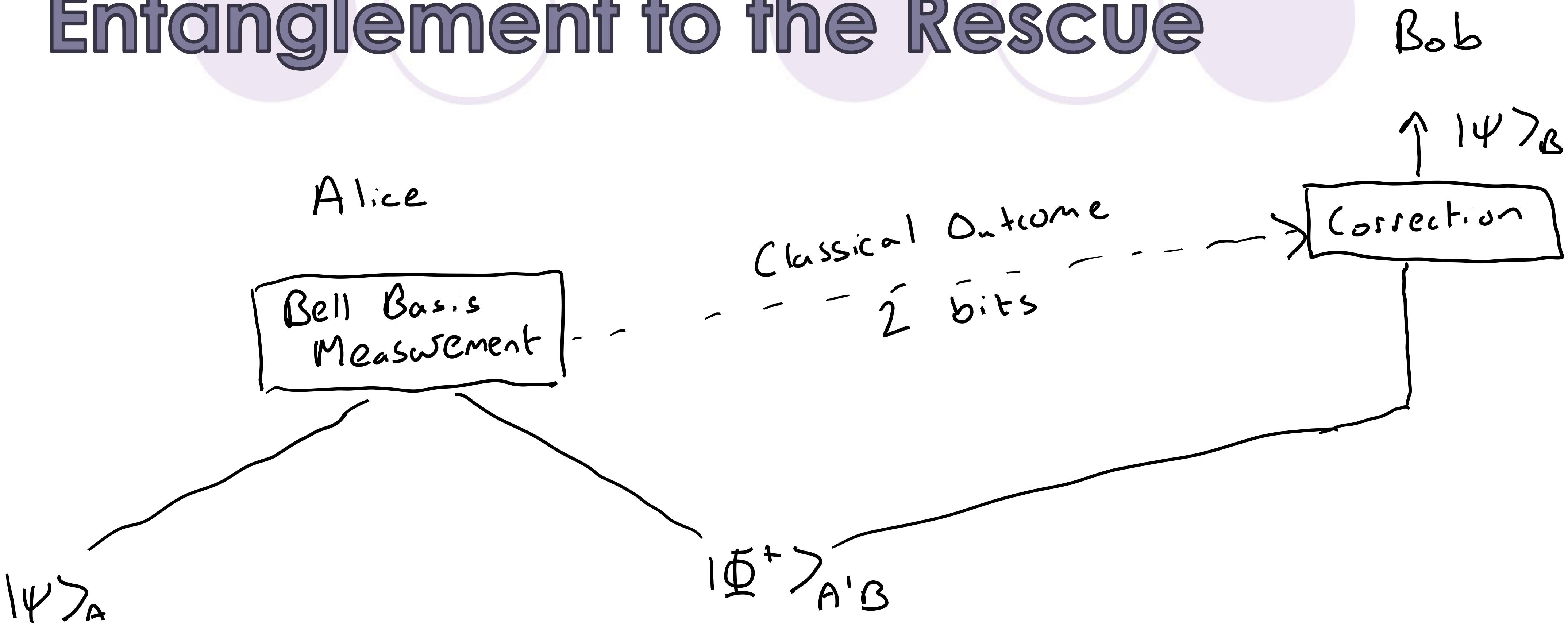
$|\psi\rangle_A$

- Suppose Alice (on Venus) has a qubit in an unknown state that she wants to send to Bob (on Mars).
- The problem is they have no communication channel through which they can reliably send quantum systems.
- They only have an old fashioned telephone line, through which they can send classical data.
- Can Alice send the state to Bob?

Quantum Teleportation

- At first sight, it seems that she obviously can't. As they only have a classical channel, she would have to convert the quantum state into classical information.
- As she does not know what the state is, she would have to measure it.
- But there is no measurement that will reliably tell her what the quantum state is (otherwise the no-cloning theorem would be violated).
- At best, she could send what she learns from the measurement, which would enable Bob to reconstruct a very unreliable approximation of $|\psi\rangle$.

Entanglement to the Rescue



- If Alice and Bob can do it if they pre-share two qubits in the entangled state

$$|\Phi^+\rangle_{A'B} = \frac{1}{\sqrt{2}}\left(|00\rangle_{A'B} + |11\rangle_{A'B}\right)$$

Quantum Teleportation Protocol

1. Alice and Bob share two qubits in the entangled state
$$|\Phi^+\rangle_{A'B} = \frac{1}{\sqrt{2}}\left(|00\rangle_{A'B} + |11\rangle_{A'B}\right)$$
2. Alice performs a joint measurement of her system A in the unknown state $|\psi\rangle_A$ and A' in the Bell basis
$$\left|\Phi^\pm\right\rangle_{AA'} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AA'} \pm |11\rangle_{AA'}\right)$$
$$\left|\Psi^\pm\right\rangle_{AA'} = \frac{1}{\sqrt{2}}\left(|01\rangle_{AA'} \pm |10\rangle_{AA'}\right)$$
3. Alice communicates the outcome to Bob. There are 4 possible outcomes, so 2 bits of communication.
4. Depending on the outcome, Bob applies one of four unitary operations to his qubit B. This transforms his system to $|\psi\rangle_B$.

Proving it Works the Old Fashioned Way

- Let $|\psi\rangle = a|0\rangle + b|1\rangle$. The initial state of the three systems is

$$|\psi\rangle_A \otimes |\Phi^+\rangle_{A'B} = (a|0\rangle_A + b|1\rangle_A) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{A'B} + |11\rangle_{A'B}\right)$$

$$= \frac{1}{\sqrt{2}}\left(a|000\rangle_{AA'B} + a|011\rangle_{AA'B} + b|100\rangle_{AA'B} + b|111\rangle_{AA'B}\right)$$

$$= \frac{1}{2\sqrt{2}}\left(|00\rangle_{AA'} + |11\rangle_{AA'}\right) \otimes (a|0\rangle_B + b|1\rangle_B)$$

$$+ \frac{1}{2\sqrt{2}}\left(|00\rangle_{AA'} - |11\rangle_{AA'}\right) \otimes (a|0\rangle_B - b|1\rangle_B)$$

$$+ \frac{1}{2\sqrt{2}}\left(|01\rangle_{AA'} + |10\rangle_{AA'}\right) \otimes (b|0\rangle_B + a|1\rangle_B)$$

$$+ \frac{1}{2\sqrt{2}}\left(|01\rangle_{AA'} - |10\rangle_{AA'}\right) \otimes (b|0\rangle_B - a|1\rangle_B)$$

Proving it Works the Old Fashioned Way

$$|\psi\rangle_A \otimes |\Phi^+\rangle_{A'B} = \frac{1}{2}(|\Phi^+\rangle_{AA'} \otimes U_0|\psi\rangle_B + |\Phi^-\rangle_{AA'} \otimes U_1|\psi\rangle_B$$
$$+|\Psi^+\rangle_{AA'} \otimes U_2|\psi\rangle_B + |\Psi^-\rangle_{AA'} \otimes U_3|\psi\rangle_B)$$

where

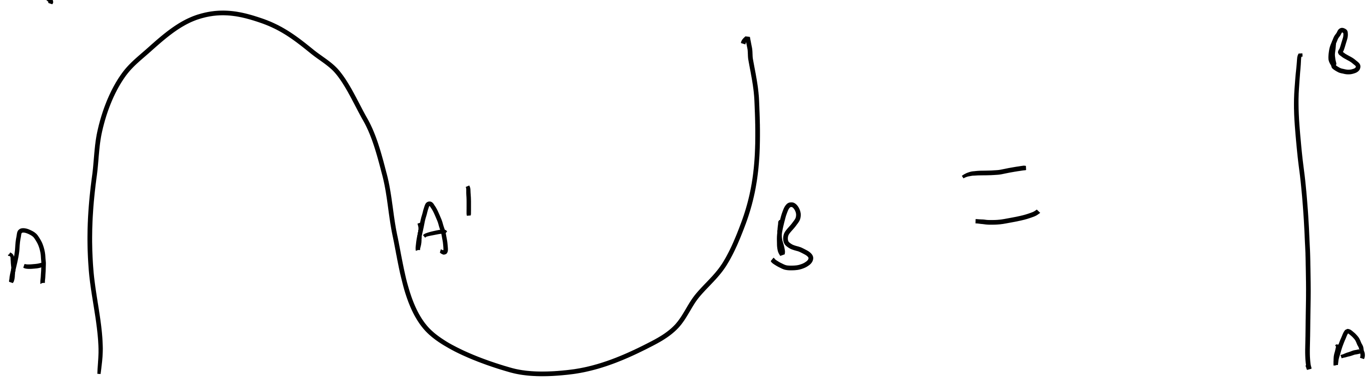
$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Now, the partial inner product with $|\Phi^+\rangle_{AA'}$ is just $\frac{1}{2}U_0|\psi\rangle_B$, which has norm $\frac{1}{4}$, so Alice will get this outcome with probability $\frac{1}{4}$. If she does, Bob just has to apply $U_0^\dagger = U_0^T$ (which is the identity in this case so he does nothing) to obtain the state $|\psi\rangle_B$.
- The same is true for the other three outcomes, so, provided Bob knows the outcome of Alice's measurement, he can apply the appropriate unitary to obtain $|\psi\rangle_B$.

Proving it Works Using Diagrams

- Recall the Yanking axiom

$\langle\delta| = \langle 00| + \langle 11|$



$|\delta\rangle = |00\rangle + |11\rangle$

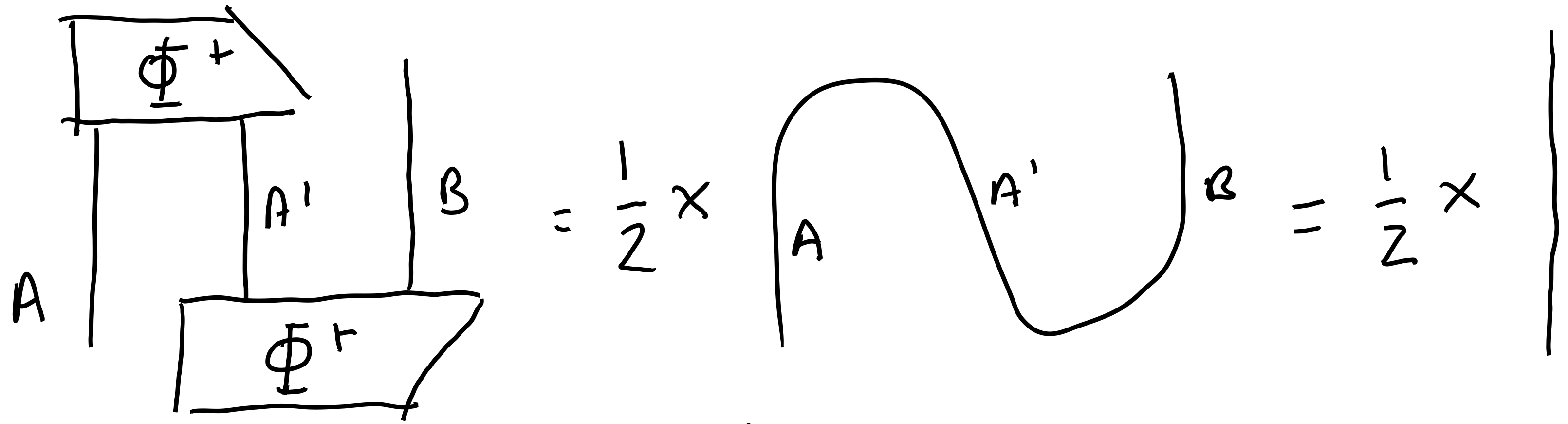
Identity $I = |0\rangle\langle 0| + |1\rangle\langle 1|$

So this axiom says

$${}_{AA'}\langle\delta|\delta\rangle_{A'B} = I_{A\to B}$$

Proving it Works Using Diagrams

- Now $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|\delta\rangle$, so this axiom tells us that:



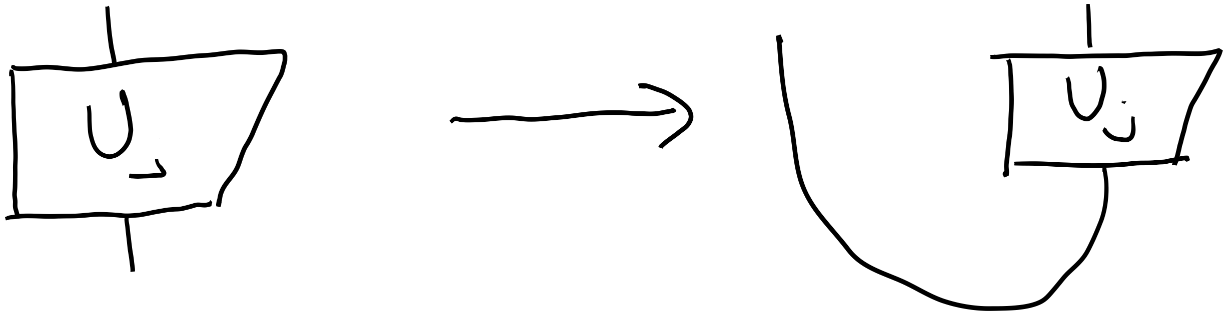
- In other words, if you prepare $|\Phi^+\rangle_{AB}$ and get the outcome$|\Phi^+\rangle_{AA'}$ in a measurement, you'll get an identity channel from A to B, up to a factor of $\frac{1}{2}$.
- The factor $\frac{1}{2}$ is just a scalar in front of the output state. It's modulus squared is the probability of this outcome happening, which is $\frac{1}{4}$.

Proving it Works Using Diagrams

- What about the other 3 outcomes. Well, the four unitary matrices U_0, U_1, U_3 are orthogonal according to the Hilbert-Schmidt inner product

$$\mathrm{Tr}\left(U_j^T U_k\right) = 2\delta_{jk}$$

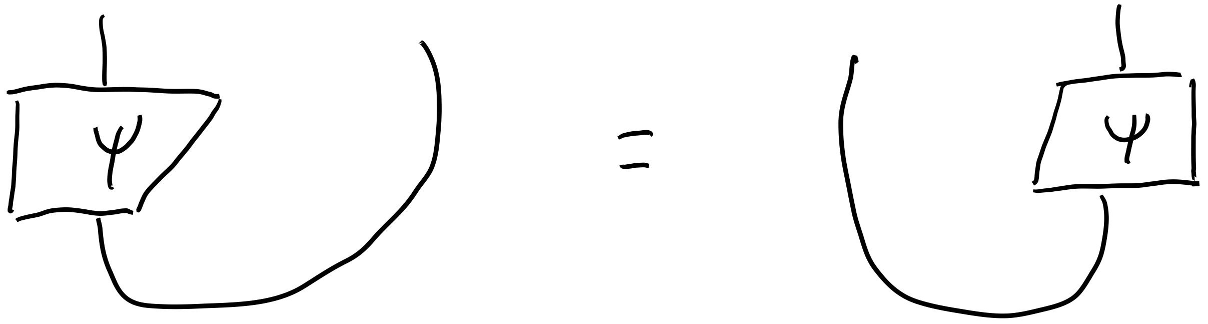
- We can convert them into states using the vector-operator correspondence:



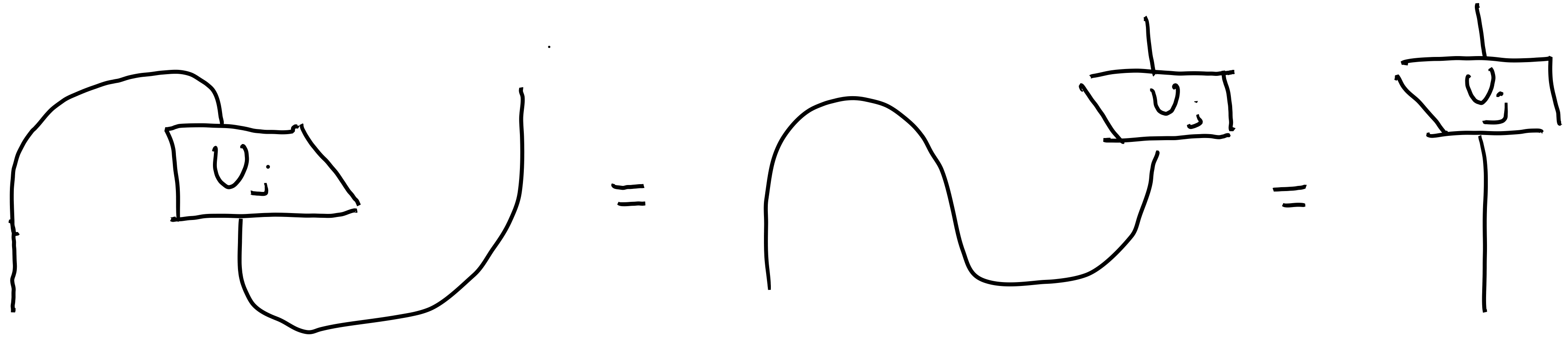
- This correspondence preserves inner products, so we will get an orthogonal basis.
- Would you believe that these are just the four states in the Bell basis (up to normalization)?

Proving it Works Using Diagrams

- So, let's see what happens when we measure in this basis. Last lecture we proved:

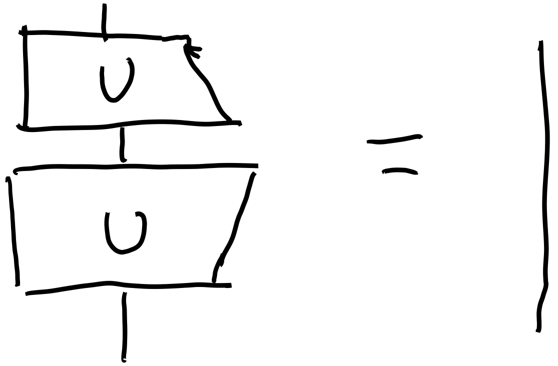


- Hence:

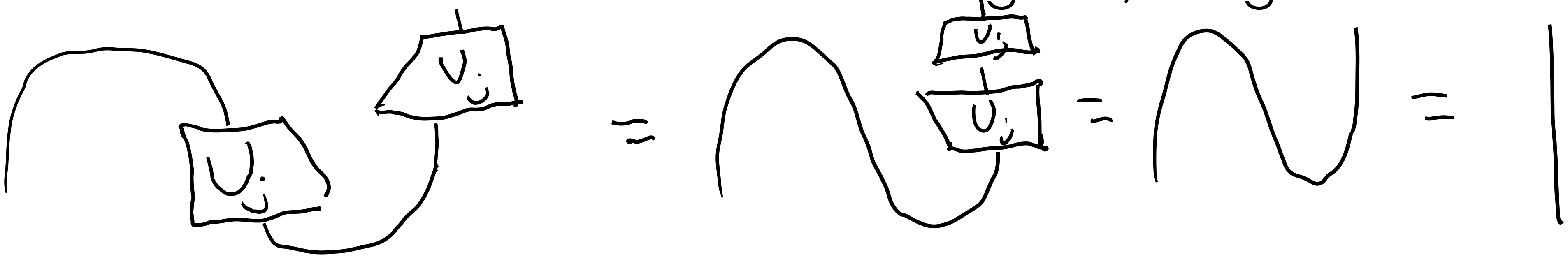


Proving it Works Using Diagrams

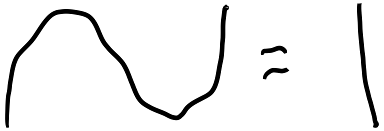
- So this shows that the total effect is to perform the transformation $U_j^* = U_j$ to the system, which Bob can undo with $\left(U_j^*\right)^\dagger = U_j^T$.
- By the way, unitarity in diagrams is expressed as



- So, if we include Bob's correction in the diagram, we get:



Is Teleportation Weird?

- Consider the ψ-ontic view of quantum states. The unknown qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ that Alice sends to Bob is specified by two complex numbers, which can take a continuum of values.
- It takes an infinite number of bits to specify these precisely, but if you are a ψ-ontologist you believe that the ontic state contains all this information, so this is really physically transmitted from Alice to Bob.
- But Alice only sends two bits of information to Bob, so how did this infinite amount of information get transmitted.
- Some people have suggested that it goes backwards in time, as suggested by taking the diagram  literally.

Classical Teleportation

- On the ψ-epistemic view of quantum states (i.e. quantum states are something more like classical probability distributions) it is not so weird.

- A classical probability distribution $\begin{pmatrix} p \\ 1-p \end{pmatrix}$ for a bit is also specified by a continuous parameter, which takes an infinite number of bits to specify.
- But Alice can transmit it to Bob by sending just one bit to Bob, i.e. just send the bit itself.

Classical Teleportation

- In fact, there is a protocol that looks a lot like teleportation:
  1. Alice and Bob share two bits that are perfectly correlated. With probability $\frac{1}{2}$ they are both 0 and with probability $\frac{1}{2}$ they are both 1.
  2. Alice has another bit with an unknown probability distribution that she wants to send to bob.
  3. Alice checks whether her two bits are the same or different. Each possibility will happen with probability $\frac{1}{2}$.
  4. If they are the same, Bob does nothing. If they are different, Bob flips his bit. Bob's bit now has the same probability distribution as Alice's original bit.

Classical Teleportation

- This classical teleportation protocol has another name: the *one-time-pad* or *Vernam cipher*. It is a way for Alice to transmit information to Bob securely if they share correlated random bits.
- Since the bit Alice sends to Bob is uniformly random, it conveys no information about the bit Alice is trying to send to an eavesdropper who does not chare Alice and Bob's correlated bits.
- The same is true of quantum teleportation. An eavesdropper learns nothing about the quantum state Alice is sending to Bob.
- In any case, on the ψ-epistemic view, if the ontic state of a qubit contains only two-bits of information and the rest of the parameters of the quantum state only express knowledge about those bits then there would be no mystery.
- Unfortunately, there are many obstacles to this idea, as we shall see later in the course.

8) The Generalized Formalism

AKA Everything I taught you in PHYS451 is wrong

i. The Two Churches of Quantum Theory
ii. The Hilbert Space of Hermitian Matrices
iii. Density Operators
iv. Completely Positive Maps
v. Positive Operator Valued Measures
vi. Quantum Instruments
vii. The Lindblad Equation

The Generalized Formalism

- In undergraduate quantum mechanics, we normally assume:
  - The system does not interact with its environment unless it is being measured.
  - Measurements are of the most ideal kind possible.
  - We have perfect knowledge of what our experimental devices are doing.
- These assumptions are never true in practice. When they do not hold, we have to generalize the formalism.
- We have already seen part of this in the GPT section: density matrices and POVMs. We will review them again, but there is much more.
- Supplementary reading for this section:
  - Teiko Heinosaari and Mario Ziman, "The Mathematical Language of Quantum Theory", Cambridge University Press (2012)
  - Benjamin Schumacher and Michael Westmoreland, "Quantum Processes, Systems, and Information", Cambridge University Press (2010)
  - Michael Nielsen and Isaac Chuang, "Quantum Computation and Quantum Information", Cambridge University Press (2000)

3.i) The Two Churches of Quantum Theory

- The Church of The Larger Hilbert Space:
  - Quantum theory is a dynamical theory, akin to a classical field theory, but with a weirder object called the wavefunction in place of a classical field.
  - All is to be derived from a quantum state (of the universe in principle) evolving unitarily according to the Schrödinger equation.
  - Today, we will allow projective measurements as well, but see lecture on Everett/many-worlds for how to derive them.
- The Church of The Smaller Hilbert Space:
  - Something strange has happened to our physical variables: they have become noncommutative.
  - Quantum theory is the only consistent probability theory for such variables.
- In this section, we will give both churches views on each construction.

3.ii) The Hilbert Space of Hermitian Matrices

- As it is a Hilbert space $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ must have multiple orthonormal bases.
- The standard basis that we have been using is just $|j,k\rangle_{AB} = |j\rangle_B \otimes {}_A\langle k|$

$$\langle jk | lm \rangle_{AB} = \left(|j\rangle_B \otimes {}_A\langle k|\right)^\dagger \left(|l\rangle_B \otimes {}_A\langle m|\right) = \langle j|l\rangle_B \langle m|k\rangle_A = \delta_j^l \, \delta_m^k$$

- Clearly, $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ has dimension $d_A \times d_B$
- But there are other bases, e.g. consider $\mathcal{L}(\mathcal{H}_A)$ with $\mathcal{H}_A = \mathbb{C}^2$ and let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $S_j = \frac{1}{\sqrt{2}} \sigma_j$ is an orthonormal basis as $\mathrm{Tr}(S_j^\dagger S_k) = \delta_{jk}$

- Consequently, every 2×2 operator can be written as $M = \frac{1}{2} \sum_j M_j \sigma_j$ with $M_j = \mathrm{Tr}(\sigma_j M)$

The Space of Hermitian Matrices

- If input and output spaces are the same, we can have Hermitian matrices

$$M^\dagger = M \quad \text{with} \quad M = \sum_{jk} M_{jk} |j\rangle_A \otimes {}_A\langle k| \qquad M^\dagger = \sum_{jk} M_{jk}^* |k\rangle_A \otimes {}_A\langle j|$$

- The set of Hermitian matrices on $\mathcal{H}_A$, denoted $\mathcal{S}(\mathcal{H}_A)$ is a Hilbert space over $\mathbb{R}$.

i.e. if $M^\dagger = M$ $N^\dagger = N$ then $(\alpha M + \beta N)^\dagger = \alpha M + \beta N$

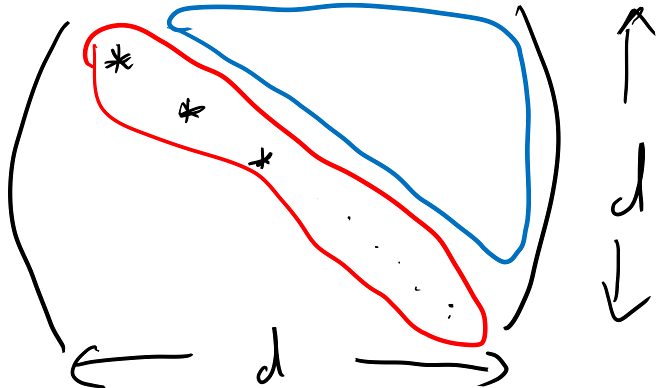
so long as $\alpha, \beta \in \mathbb{R}$

and $\mathrm{Tr}(M^\dagger N) \in \mathbb{R}$

- The dimension of this space is d^2

d real parameters $+$ $(d-1)d$ real parameters

$= d^2$ real parameters



Hermitian Bases

- The matrices $|j\rangle\langle k|$ are not Hermitian, but there must be a basis of d^2 Hermitian matrices.
- We already saw $\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

  which is a Hermitian basis for operators on $\mathbb{C}^2$
- In general, you can take

$$\left.\begin{array}{l}\frac{1}{\sqrt{2}}\left(|j\rangle\langle k| + |k\rangle\langle j|\right)\\ \frac{1}{\sqrt{2}}\left(i|j\rangle\langle k| - i|k\rangle\langle j|\right)\end{array}\right\}$$

These are not all different and not all nonzero (e.g. take $j=k$)

Proper counting gives d^2 orthonormal matrices.

Hermitian Matrices are Self-Dual

- Because $\mathcal{S}(\mathcal{H}_A) \subseteq \mathcal{L}(\mathcal{H}_A) \cong \mathcal{H}_A \otimes \mathcal{H}_A^\dagger$, the dual $\mathcal{S}(\mathcal{H}_A)^\dagger$ won't be all of $\mathcal{L}(\mathcal{H}_A)$
- Because $\mathcal{S}(\mathcal{H}_A)$ is a real Hilbert space, $\mathcal{S}(\mathcal{H}_A)^\dagger$ consists of linear functionals from $\mathcal{S}(\mathcal{H}_A)$ to $\mathbb{R}$, not $\mathbb{C}$
- Because $\mathcal{S}(\mathcal{H}_A)$ is a Hilbert space, the inner product still induces an isomorphism $\mathcal{S}(\mathcal{H}_A) \cong \mathcal{S}(\mathcal{H}_A)^\dagger$

Write $M \in \mathcal{S}(\mathcal{H}_A)$ in terms of a self-adjoint basis

$$M = \sum_j m_j N_j$$

real coefficients

Then $M^\dagger = \sum_j m_j N_j^\dagger = \sum_j m_j N_j$ so a vector in $\mathcal{S}(\mathcal{H}_A)$ is its own dual vector

The Space of Commutative Matrices

- Consider a maximal set of commuting matrices on $\mathcal{H}_A$, i.e. the set of operators that are diagonal in a common basis. This is a Hilbert space (over $\mathbb{C}$)
- If $|j\rangle$ is the diagonalizing basis then $|j\rangle\langle j|$ is a matrix basis, since all matrices can be written as

$$M = \sum_j \lambda_j |j\rangle\langle j|$$

dimension is d.

eigenvalues

- If we restrict attention to Hermitian commuting operators (λ_j's real) then this is also a Hilbert space, now over $\mathbb{R}$. Denote this space as $\mathcal{C}(\mathcal{H}_A)$. (Matrices in $\mathcal{C}(\mathcal{H}_A)$ are also their own duals)

3.iii) Density Operators

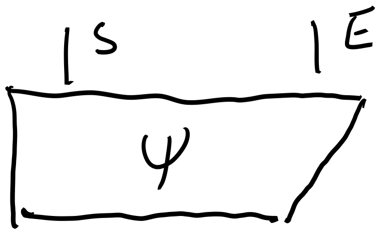
- According to the larger church, the universe always has a pure state vector $|\psi\rangle$.
- If any other mathematical object is used for a quantum state, it must be because we are looking at a subsystem.
- State space is $\mathcal{H}_S \otimes \mathcal{H}_E$

System we are interested in

Environment

and we have $|\psi\rangle_{SE} = \sum_{jk} \psi^{jk} |j\rangle_S \otimes |k\rangle_E$ or $\psi^{j_S k_E}$

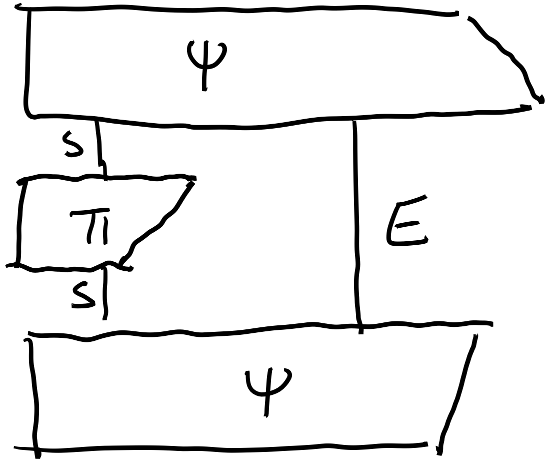
or

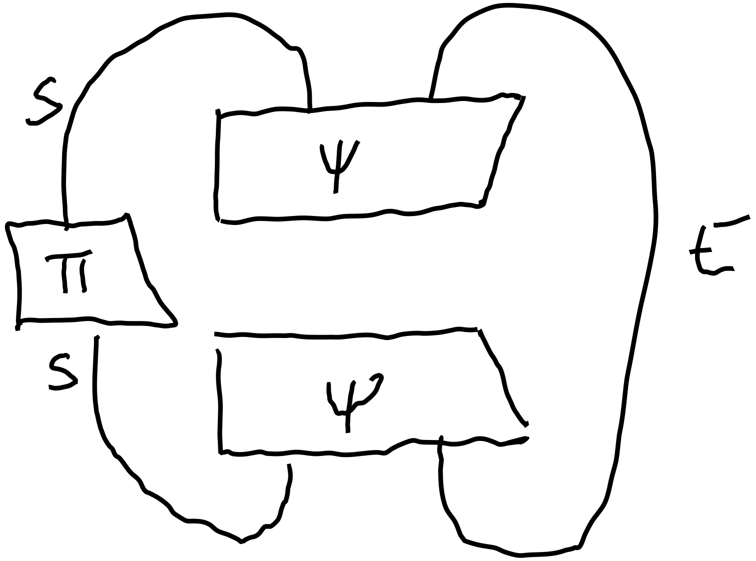


The View from the Larger Church

① Suppose we make a projective measurement on system S alone. The probability of getting outcome corresponding to projector Π is

$${}_{SE}\langle\psi|\,\Pi_S\otimes I_E\,|\psi\rangle_{SE} = \psi^{\dagger}_{j_S k_E}\,\Pi^{j_S}_{l_S}\,\psi^{l_S k_E} =$$





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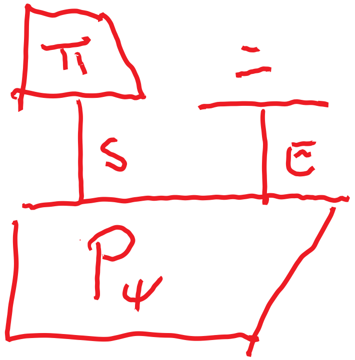
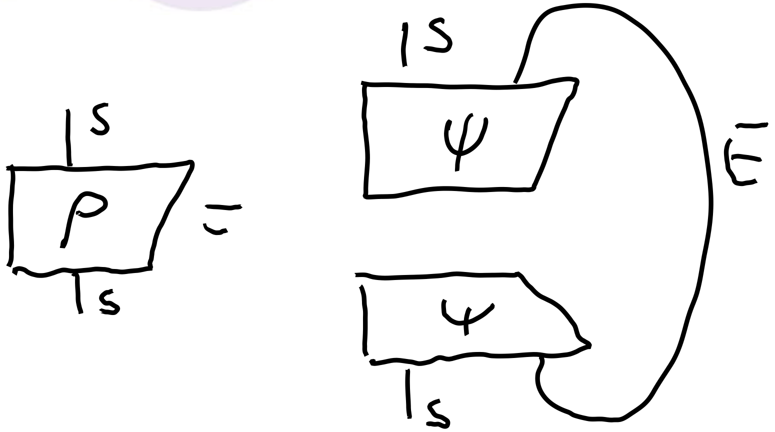


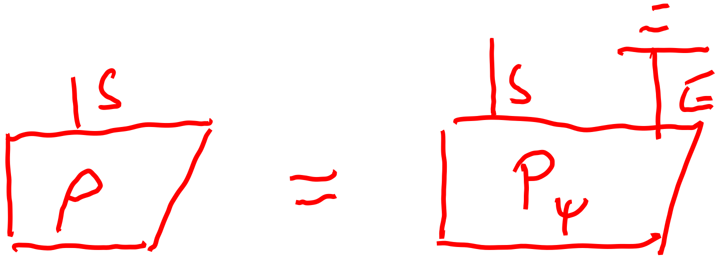
where $P_\psi = |\psi\rangle\langle\psi|$

The View From The Larger Church

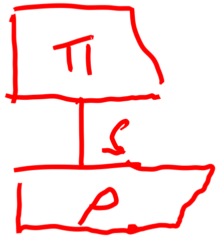
- If we define the object

$$\rho_S = \mathrm{Tr}_E(|\psi\rangle\langle\psi|) \qquad \rho^{i_S}_{\ j_S} = \psi^{i_S k_E}\, \psi^{\dagger}_{j_S k_E}$$





then the probability is $\mathrm{Prob}(\Pi) = \mathrm{Tr}(\Pi_S \rho_S) = \Pi^{k_S}_{\ j_S}\, \rho^{j_S}_{\ k_S} =$



- ρ lives in the space $\mathcal{H}_S \otimes \mathcal{H}_S^{\dagger} \equiv \mathcal{L}(\mathcal{H}_S) \equiv \mathcal{L}(\mathcal{H}_S)^{\dagger}$

  so it is both an operator and a duperator

- We normally call it a <u>density operator</u> (although we use it as a duperator)