# Quantum roundations Lecture 11 

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Dr. Matthew Leifer leifer@chapman.edu HSC112

## 7) Abstract Tensor Systems

i. Vectors, Dual Vectors, Inner Products, and Tensor Products
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## 7) Abstract Tensor Systems

- The aim of this section is to introduce a way of dealing with systems with complicated tensor products using diagrams.
- This can be used to visualize or simplify various proofs in the next section.
- As an application, we will discuss quantum teleportation.
- For much more on this approach, see Bob Coecke and Aleks Kissenger, "Picturing Quantum Processes", Cambridge University Press (2017) in Supplemetal Reading.
7.iii) Abstract Index Notation
$\bigcirc$ It is cumbersome to keep track of long strings of bras/kets

$$
\left.e \cdot g . \quad|j\rangle_{A} \otimes \mid h\right)_{B} \otimes|l\rangle_{C} \otimes \ldots
$$

- We can develop an abstract index notation similar to that used in differential geometry and $G R$.

$$
|\psi\rangle_{A}=\left.\sum_{j} \psi^{j}\right|_{j\rangle_{A}} \Rightarrow \psi^{j_{A}} \quad\langle\phi|=\sum_{j} \phi_{j} j_{A} \mid \Rightarrow \phi_{j_{A}}
$$

$\langle\phi \mid \psi\rangle_{A}=\phi_{j_{A}} \psi^{j_{A}} \leftarrow$ summation convention for repeated indices

- It is necessary to include the label $A$ of the Hilbert space $\mathcal{H}_{A}$ in the index $j_{A}$ because Hilbert spaces may have different dimensions Only upper $A$ indices can be contracted with lower $A$ indices.

Abstract index Notation
O For a tensor product space $H_{A} \otimes \mathcal{H}_{B}$, we would have

$$
\begin{aligned}
& |\psi\rangle_{A B}=\sum_{j k} \psi^{j h}|j\rangle_{A} \otimes|h\rangle_{B} \Rightarrow \psi^{j A k_{B}} \\
& \Delta \delta \phi \mid=\sum_{j k} \phi_{j k}\left\langle_{A}\right| \otimes\left\langle_{B} h\right| \Rightarrow \phi_{j A B}
\end{aligned}
$$

$\bigcirc$ The inner product is

$$
\langle\phi \mid \psi\rangle_{A B}=\phi_{j_{A} k_{B}} \psi^{j_{A} k_{B}}
$$

O However $\phi_{j_{A} h_{B}} \psi^{k_{A} j_{B}}$ is not a valid contraction

## 7.iiii) Diagrammatic Nofation

- Even abstract tensors get tedious after a while, so it is useful to develop a way of representing them with diagrams.
- A tensor is represented by a box.
- A vector index is represented by an upward directed line with Hilbert space label.
- A dual vector index is represented by a downward directed line with a Hilbert space label.
- Contraction (taking inner products) is represented by joining lines.

Examples

$$
\begin{aligned}
& |\psi\rangle_{A}=\sum_{j} \psi^{j}|j\rangle_{A} \Leftrightarrow \psi^{j A} \Longleftrightarrow \psi^{A} \\
& \langle\phi|=\sum_{j} \phi_{j A}\langle j| \Longleftrightarrow \phi_{j A} \Longleftrightarrow \frac{\phi}{1 A} \\
& \langle\phi \mid \psi\rangle_{A}
\end{aligned} \Longleftrightarrow \phi_{j A} \psi^{j_{A}} \Longleftrightarrow \frac{\phi}{1 A}
$$

Examples

$$
\begin{aligned}
& \left.|\psi\rangle_{A B}=\left.\sum_{j k} \psi^{j k}\right|_{j}\right\rangle_{A} \otimes|k\rangle_{B} \Leftrightarrow \psi^{j_{A} k_{B}} \Longleftrightarrow \\
& { }_{A B}\langle\phi|=\sum_{j h} \phi_{j k A j \mid \otimes}\langle h| \Leftrightarrow \phi_{j, k B B} \Leftrightarrow \phi_{T_{A} T_{B}} \\
& \langle\phi \mid \psi\rangle_{A B} \Leftrightarrow \phi_{j_{A} k_{B}} \psi^{j_{A} k_{B}} \Longleftrightarrow \frac{\phi}{\psi}
\end{aligned}
$$

Diagrammatic $\mathbb{N}$ ofation

- Note: The shape of a box does not matter. Only the direction of the lines coming out of it matters, e.g.

7.iv) More Interesting Tensor Products
$\bigcirc \mathcal{H}_{A}$ and $H_{B}^{+}$are both Hilbert spaces, so there is no reason why we can't form the tensor product $\mathcal{H}_{A} \otimes \mathcal{H}_{B}^{+}$
O This would be the vector space of objects of the form

$$
\sum_{j h} \psi_{h}^{j}|j\rangle_{A} \otimes\langle h| \Leftrightarrow \psi_{h_{B}}^{j a} \Leftrightarrow
$$


$\bigcirc$ The dual space to $\mathcal{H}_{A} \otimes \mathcal{H}_{B}^{+}$is $\left(\mathcal{H}_{A} \otimes \mathcal{H}_{B}^{\dagger}\right)^{\dagger}=\mathcal{H}_{A}^{+} \otimes \mathcal{H}_{B}$

$$
\sum_{j k} \phi_{j}^{k}\left\langle_{A}\right| \otimes|k\rangle_{B} \Leftrightarrow \frac{\phi_{j A}^{k_{B}}}{1_{A}}=\frac{1^{B}}{\phi \square_{B}^{B}}
$$

$O$ The inner product is given by $\phi_{j_{A}}^{k_{B}} \psi_{k_{B}}^{j_{A}} \Longleftrightarrow$

More Interesting Tensor Products
○ An object like $\sum_{j k} \psi_{k}^{j}|j\rangle_{A} \otimes\langle h|$ is neither a kit nor a bra.
$\bigcirc$ However $\mathcal{H}_{A} \otimes \mathcal{H}_{B}^{\dagger}$ is still just a Hilbert space, like any other.
O Sometimes it will be useful to think of $\mathcal{H}_{A} \otimes \mathcal{H}_{B}^{+}$as a space of "kets" and its dual $\mathcal{H}_{A}^{+} \otimes \mathcal{H}_{B}$ as a space of "bras".
OWen doing so, 1 will use red brakets and red diagrams

$$
|\psi\rangle_{A B}=\sum_{j k} \psi_{k}^{j}|j\rangle_{\Lambda} \otimes\left\langle_{B} k\right|
$$



More interesting Tensor Products
O Clearly, we can iterate this construction and consider complicated tensor products egg. $H_{L}=\mathcal{H}_{A} \otimes \mathcal{H}_{B}^{+} \otimes \mathcal{H}_{C}^{+} \otimes \mathcal{H}_{D} \otimes \mathcal{H}_{E}^{+}$

$$
\left.\sum_{j k \mid m n} \psi_{k l n}^{j m}|j\rangle_{A} \otimes_{B}\langle k| \otimes_{C}\langle ||\otimes| m\right\rangle_{D} \otimes_{E}\langle n| \Leftrightarrow \psi_{k_{B} l_{C} n_{E}}^{j_{A} n_{D}} \Leftrightarrow
$$


$O$ It's dual space is $\mathscr{H}^{+}=\mathcal{H}_{A}^{+} \otimes \mathcal{H}_{B} \otimes \mathcal{H}_{C} \otimes \mathcal{H}_{D}^{+} \otimes \mathcal{H}_{E}$

$$
\sum_{j k(m n} \phi_{j m}^{k i n}\langle j| \otimes|k\rangle
$$

$$
\phi_{j_{A} M_{D}}^{k_{B} l_{C} n_{E}} \psi_{k_{B} l_{C} n_{E}}^{J_{A} m_{B}} \Longleftrightarrow
$$

Partial inner Products
O We can also define partial inner products where we only contract over some of the indicies
e.g. $\quad \psi_{k_{B}}^{j_{A}} l_{c} \in \mathcal{H}_{A} \otimes \mathcal{H}_{B}^{+} \otimes \mathcal{H}_{C} \quad \phi_{L_{C} m_{0}}^{k_{B}} \in \mathcal{H}_{B} \otimes \mathcal{H}_{C}^{+} \otimes \mathcal{H}_{D}^{+}$

7.v) The Space of Linear Operators

O Now let's consider the space of linear operators from $H_{A}$ to $\mathcal{H}_{B}$ denoted $\mathcal{L}\left(\mathcal{H}_{A} \rightarrow \mathcal{H}_{B}\right) \quad\left[\right.$ just $\mathcal{L}\left(\mathcal{H}_{A}\right)$ it $\left.\mathcal{H}_{A}=\mathcal{H}_{B}\right]$
O In Dirac notation, we know that an operator can be written in terms of its matrix elements

$$
\left.\left.M=\left.\sum_{j h} M_{h}^{j}\right|_{j}\right\rangle_{B}\left\langle_{A} h\right| \text { where } M_{h}^{j}=\widehat{B}_{\beta}|M| h\right\rangle_{A}
$$

O But this looks just like an object in $\mathcal{H}_{B} \otimes \mathcal{H}_{A}^{\dagger}$

$$
\sum_{j k} M_{k}^{j}|j\rangle_{B} \otimes \lambda_{A} k \Leftrightarrow M_{h_{A}}^{j_{B}} \Leftrightarrow \frac{B}{M_{A}}
$$

The Space of Linear Operators

- If we treat $M$ as an element of $\mathcal{H}_{B} \otimes \mathcal{H}_{A}^{\dagger}$ then the action of $M$ on a vector $|\psi\rangle_{A} \in X_{A}$ is just partial inner product

$$
\begin{aligned}
& M=\sum_{j n} M_{n}^{j}|j\rangle_{B} \otimes\left\langle_{A} H\right| \quad|\psi\rangle_{A}=\sum_{l} \psi^{l}|l\rangle_{A} \\
& \left.M|\psi\rangle_{A}=\left.\sum_{j k} M_{h}^{j} \psi^{k}\right|_{j}\right\rangle_{B} \quad \text { or } \quad M_{n_{B}}^{j_{A}} \psi^{k_{B}}
\end{aligned}
$$



The Space of Linear Operators
$\bigcirc$ In general, the space of linew operators from $\mathcal{H}_{A}$ to $\mathcal{H}_{B}$ is (isomorphic to)

$$
\mathcal{H}_{B} \otimes \mathcal{H}_{A}^{+} \equiv \mathcal{L}\left(\mathcal{H}_{A} \rightarrow \mathcal{H}_{B}\right)
$$

$$
\text { output space } \hat{\mathrm{h}}^{\text {dual }} \text { of input space }
$$

O Everything can be done with tensor products and partial inner products!
$\bigcirc$ Note, the space $\mathcal{H}_{B} \otimes \mathcal{H}_{A}^{+} \equiv \mathcal{L}\left(\mathcal{H}_{A} \rightarrow \mathcal{H}_{B}\right)$ is just what we introduced the red bets and diagrams for

$$
|\psi\rangle_{A B} \in \mathcal{H}_{B} \otimes \mathcal{H}_{A}^{+}
$$

$$
\frac{\frac{1 B}{\psi}}{T_{A}}=\frac{\mid A B}{\psi}
$$

If $\mathcal{H}_{A}=\mathcal{H}_{B}, 1$ will abbreviate $|\psi\rangle_{A} \in \mathcal{H}_{A} \otimes \mathcal{H}_{A}^{+}$


Duperafors
O The dual of an operator is a linew functional from operators to scalars (a "duperator")
$O\left(\mathcal{H}_{B} \otimes \mathcal{H}_{A}^{+}\right)^{+}=\mathcal{H}_{A} \otimes \mathcal{H}_{B}^{+} \quad$ so $\quad \mathcal{L}\left(\mathcal{H}_{A} \rightarrow \mathcal{H}_{B}\right)^{+}=\mathcal{L}\left(\mathcal{H}_{B} \rightarrow \mathcal{H}_{A}\right)$
ie. The duperators from $A$ to $B$ are the operators from $B$ to $A$ Operator from $A$ to $B$ :

$$
\begin{aligned}
& \text { rotor from } A \text { to } B: \\
& |\psi\rangle_{A B}=\left.\sum_{j k} \psi_{j}^{k}\right|_{j\rangle_{A} \otimes} \leqslant k \mid
\end{aligned} \quad \psi_{j A}^{k_{B}} \Leftrightarrow \frac{\left.\right|_{A}}{I_{A}} \Leftrightarrow \psi
$$

Duporator from $A$ to $B$ :

$$
{ }_{A B}\langle\phi|=\sum_{j h} \phi_{k}^{j}\langle j| \Theta|k\rangle_{B} \Leftrightarrow \phi_{k B}^{j A} \Leftrightarrow \phi_{B}^{\mid A} \Leftrightarrow \phi_{A B}
$$

Inner Products of Operators
O Using the general correspondence between vectors and duals, we have

$$
\begin{aligned}
& |\psi\rangle_{A B}=\sum_{j h} \psi_{j}^{k}|j\rangle_{A} \otimes{ }_{B} k \left\lvert\, \Leftrightarrow \psi_{j_{A}}^{k_{B}} \Leftrightarrow \frac{\psi_{A}}{T_{A}} \Leftrightarrow \psi^{1 B}\right. \\
& \langle\psi|=\sum_{j h} \psi_{k A}^{\dagger j}\langle j| \otimes|k\rangle_{B} \Leftrightarrow \psi_{k_{B}}^{j_{A}} \Leftrightarrow \frac{\psi^{+}}{l_{B}} \Leftrightarrow \frac{\psi^{+}}{\left.\right|_{A B}}
\end{aligned}
$$ © The inner product is then:

$$
\langle\phi \mid \psi\rangle_{A B}=\operatorname{Tr}\left(\phi^{+} \psi\right)=\phi_{k_{B}}^{+j_{A}} \psi_{j_{A}}^{k_{B}} \Leftrightarrow \phi^{+} \psi^{l_{B}} \Leftrightarrow
$$

OThis is called the "Hilbert-Schmidt" inner product, and you proved it is an inner product in Hah 1.
7.vi) Raising and Lowering Indices
$\Theta$ Consider the vector $\left.|\delta\rangle_{A A}=\sum_{j}|j\rangle_{A} \otimes|j\rangle_{A}=\left.\sum_{j h} \delta^{j k}\right|_{j}\right\rangle_{A} \otimes|k\rangle_{A}$ where $\delta^{j h}=\left\{\begin{array}{l}1, j=h \\ 0, j \neq h\end{array} \quad\right.$ This lives in $\mathcal{H}_{A} \otimes \mathcal{H}_{A}$

- Abstract index notation: $\delta^{j_{A} k_{A}}$

ODingramatic notation:


O partial inner product with this twas a bra into a Ret

$$
\delta^{j_{A} k_{A}} \psi_{k A}=\psi^{j_{A}}
$$



Raising and Lowering Indices

- Similarly the dual vector $A_{A}\langle\delta|=\sum_{j}\langle j| \otimes_{A} j \mid=\sum_{j k} \delta_{j k}\left\langle K_{A}\right| \otimes_{A}\langle k|$ twins hats into bras
O Abstract notation $\delta_{j_{A} k_{A}}$
O Diagramatic notation

$$
\delta_{j_{A} h_{A}} \psi^{k_{A}}=\psi_{j_{A}}
$$




O The identity operator is $\left.I_{A}=\sum_{j}|j\rangle_{A}^{\otimes_{A}}\langle j|=\left.\sum_{j k} \delta_{k}^{j}\right|_{j}\right\rangle_{A} \otimes \Delta_{A}\langle k|$
O In abstract index notation, this is just $\delta_{k_{A}}^{j_{n}}$
O And as a diagram it is just a vertical piece of wire

$$
\frac{I^{A}}{l_{A}}=\left.\right|^{A}
$$

The Yanking Axioms
$\bigcirc$ The various $\delta$ tensors satisfy the fallowing properties


$$
\delta^{j k_{A}} \delta_{k_{A} M_{A}}=\delta_{m_{A}}^{j j_{A}}=\delta_{m_{A} k_{A}} \delta^{k_{A} j_{A}}
$$

The yanking axioms allow as to prove lots of things using just diagrams.
$\bigcirc=$


Just expresses the tact that order of indices is unimportant in abstract index notation.
7.viil) Transpose, Conjugate, Duals, and Trace
O The transpose is defined as

$$
\psi_{j_{A}}^{T k_{B} l_{c}}=\psi_{j_{A}}^{k_{B} l_{c}}=\delta_{j_{A} m_{A}} \delta^{k_{B} n_{B}} \delta^{l_{C} r_{C}} \psi_{n_{B} l_{c}}^{m_{A}}
$$



A Bit of Diogramnnatic Trickery
O We can make more intuitive diagrams if we introduce a bit of asymmetry to our boxes.


O Then we can represent transpose by $180^{\circ}$ rotation


Fun With Diagrams
O Now let's actually prove something with diagrams


O Proof: Using the Yanking axioms


Conjugate
O The conjugate of a tensor is just the tensor you obtain by taking the complex conjugate of all of its components

$$
\begin{gathered}
\sum_{j k l} \psi_{k \mid}^{j}|j\rangle_{A} \otimes{\underset{B}{ }}^{k} \mid \otimes\langle l| \longleftrightarrow \sum_{j h l}^{*}\left(\psi_{h l}^{j}\right)^{*}|j\rangle_{A} \otimes_{B}\langle k| \otimes<l l \\
\left(\psi^{*}\right)_{k_{B} l c}^{j_{A}}=\left(\psi_{k_{B} l c}^{j_{A}}\right)^{*}
\end{gathered}
$$

© In a diagram we represent it by reflecting in a vertical axis


Dual//Adjoint
O The dual or adjoint is defined as taking the complex conjugate, Followed by the transpose, or vice versa.

$$
\begin{aligned}
& \left.\sum_{j k l} \psi_{k l}^{j}|j\rangle_{A} \otimes_{B}|k| \otimes<l\left|\longleftrightarrow \sum_{j k l} \psi_{j}^{* k l}\langle j| \otimes\right| k\right\rangle_{B} \otimes|l\rangle_{c} \\
& \left(\psi^{\dagger}\right)_{j_{A}}^{k_{B} l_{c}}=\delta_{j_{A} m_{A}} \delta^{k_{B}^{n_{B}}} \delta^{l_{c} r_{c}}\left(\psi^{*}\right)_{n_{B} r_{c}}^{m_{A}}
\end{aligned}
$$

OFor obvious reasons, in diagrams it is represented by a reflection in the horizontal axis


Summary

7.vilii) Trace and Parifill Trace

O The trace of an operator $\sum_{j k} \psi_{k}^{j}|j\rangle_{A} \otimes\langle k|$ is defined as

$$
\sum_{j} \psi_{j}^{j}
$$

$O$ in abstract index notation $\psi_{j_{A}}^{j_{A}}=\delta_{j_{A} m_{A}} \psi_{k_{A}}^{m_{A}} \delta^{j_{A} k_{A}}=\delta_{m_{A} k_{A}} \psi_{j_{A}}^{m_{A}} \delta^{j_{A} k_{A}}$ In red diagrams, we will use


POTMTO] TrOMP
O We can obviously contract any indices that have the same system label. This is called a partial trace in quantum theory

$$
\begin{aligned}
\text { e.g. } & \operatorname{Tr}_{A}\left(\sum_{j k \mid m} \psi_{k m}^{j l}|j\rangle_{A} \otimes_{A}\langle k| \otimes|l\rangle_{B} \otimes\langle m|\right) \\
= & \sum_{j l m} \psi_{j m}^{j l}|l\rangle_{B} \otimes{ }_{B}\langle m|
\end{aligned}
$$



Vector Operator Correspondence
$O$ Raising and lowering indices induces a correspondence between operators in $\mathcal{L}\left(\mathcal{H}_{A} \rightarrow \mathcal{H}_{B}\right)=\mathcal{H}_{A}^{+} \otimes \mathcal{H}_{B}$ and vectors in $\mathcal{H}_{A} \otimes \mathcal{H}_{B}$

$$
\begin{gathered}
\left.\sum_{j k} \psi_{j}^{k}|k\rangle_{B} \otimes{ }_{A}\langle j| \longleftrightarrow \sum_{j k} \psi^{j k}\left|j \nu_{A} \otimes\right| k\right\rangle_{B} \\
\psi^{j a_{A B}}=\delta^{j_{A} m_{A}} \psi_{m_{A}}^{k_{B}} \quad \psi_{j_{B}}^{k_{B}}=\delta_{j_{A} m_{A}} \psi^{j_{A} j_{B}}
\end{gathered}
$$



Getting Rid of Awkward Boxes
O For vectors in mixed tensor products of bra and Ket spaces eg. $\mathcal{L}_{A} \otimes \mathcal{L}_{B}^{+}$, we had to use awtewardly shaped boxes to express the inner product as a diagram
O Using transposes and conjugates, we can now get rid of the awkward boxes


