Quantum Foundations Lecture 4

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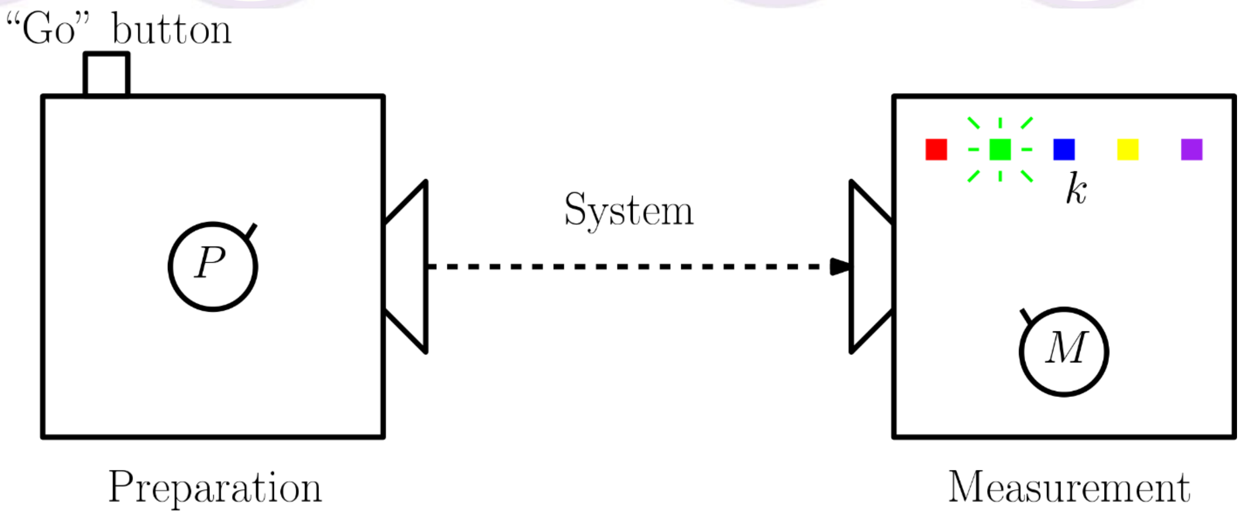
HSC112

3) Generalized Probabilistic Theories

i. Prepare-and-Measure Experiments
ii. Test Spaces
iii. States
iv. Effects
v. Rebits
vi. Qubits
vii. Quantum Theory as a GPT

3.i) Prepare-and-Measure Experiments



- We will consider a simple type of black-box experiment.
  - P is the choice of preparation setting
  - M is the choice of measurement setting
  - k is the outcome of the measurement (takes a finite number of values)
  - A *theory* must predict $\text{Prob}(k|P, M)$ for all choices of P and M.
  - *System* means whatever accounts for the correlation between P *and* (M, k).

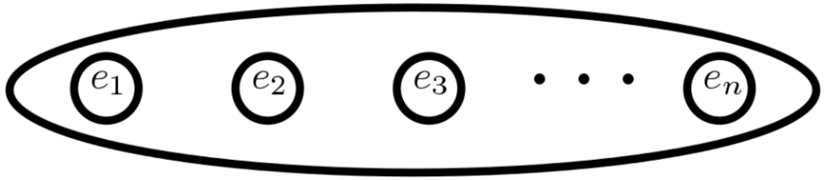
3.ii) Test Spaces

- A test space is a way of formalizing the description of a measurement device.
- A *(locally finite) test space* (X, Σ) consists of
  - A set X of *outcomes* (not necessarily finite).
  - A set Σ of *tests* (not necessarily finite).
  - Each test E is a finite subset of X, interpreted as the set of outcomes for a measurement that can be performed on the system.
- A test space is called *finite* if X is a finite set. A finite test space is also called a *hypergraph*.

Examples

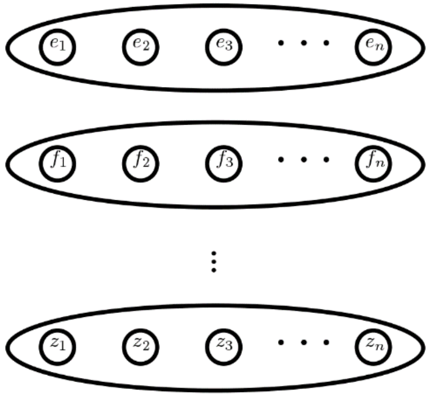
- A *classical* test space has just one test.

$(\{e_1, e_2, \cdots, e_n\}, \{\{e_1, e_2, \cdots e_n\}\})$



- A *semi-classical* test space has non-overlapping tests.

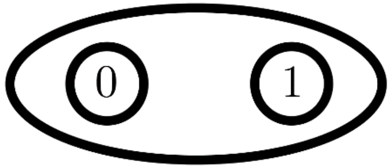
$(\{e_1, e_2, \cdots, e_n, f_1, f_2, \cdots, f_n, \cdots, z_1, z_2, \cdots, z_n\}, \{\{e_1, e_2, \cdots e_n\}, \{f_1, f_2, \cdots, f_n\}, \cdots \{z_1, z_2, \cdots, z_n\}\})$



Examples

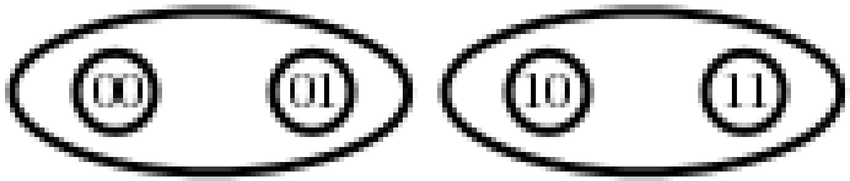
- A classical *bit*:

$$(\{0,1\},\{\{0,1\}\})$$



- A *generalized bit (gbit)* or *square bit (squit)*:

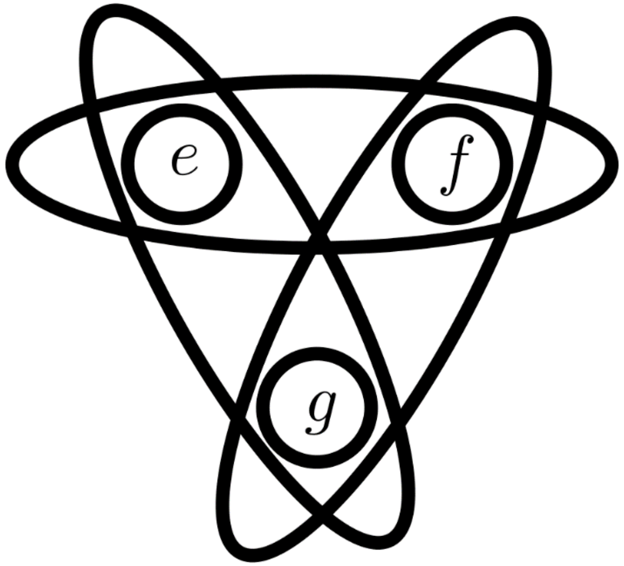
$$(\{00,01,10,11\},\{\{00,01\},\{10,11\}\})$$



Examples

- Specker's triangle:

$$(\{e, f, g\}, \{\{e, f\}, \{f, g\}, \{g, e\}\})$$



  - Importantly, tests can overlap. e, f and g each appear in two tests.
  - When the same outcome appears in more than one test, we regard it as physically the same thing in each test, perhaps because it always has the same probability for every preparation.

3.iii) States on Test Spaces

- A *state* on a test space is a function $\omega: X \to [0,1]$ such that
$$\forall E \in \Sigma, \qquad \sum_{e \in \mathrm{E}} \omega(e) = 1$$
- Each test is assigned a probability distribution and if an outcome appears in more than one test, it receives the same probability in each one.
- We can represent states as vectors $\boldsymbol{\omega} \in \mathbb{R}^X$.
  - Note: $\mathbb{R}^X$ just means $\mathbb{R}^{|X|}$, with the axes labelled by elements of X.
  - The components are given by $\omega_e = \omega(e)$.

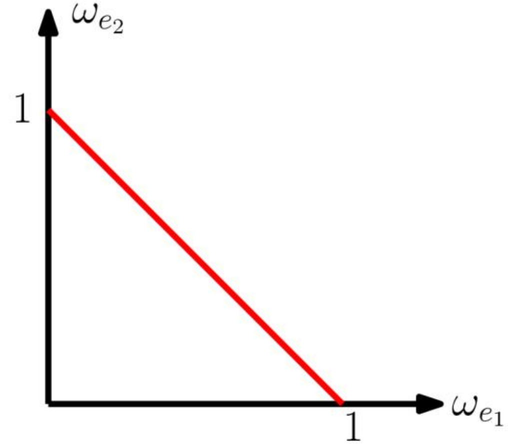
3.iii) States on Test Spaces

- The *state space* Ω of a test space is the set of all states.
  - It is defined by a set of linear inequalities
  $$\omega(e) \geq 0$$
  - Together with a set of linear equalities
  $$\sum_{e \in E} \omega(e) = 1$$
- It will therefore be a closed convex set in $\mathbb{R}^X$, but its dimension may be much less than $|X|$ because independent equalities can be used to reduce the dimension.
- If the test space is finite then the state space is a polytope.

Examples

- For a classical test space, the state space is just the probability simplex Δ_X.

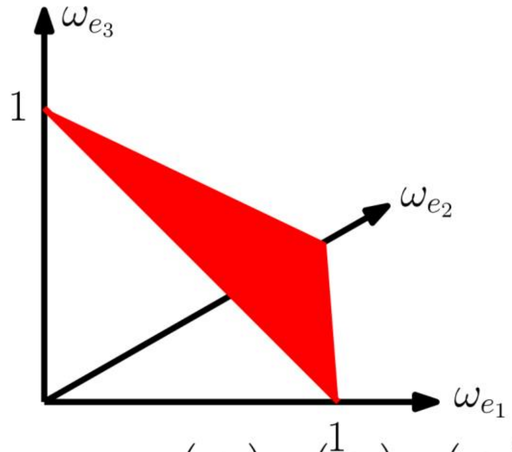
$$(\{e_1, e_2, \cdots, e_n\}, \{\{e_1, e_2, \cdots e_n\}\}) \qquad \omega(e_n) \geq 0, \qquad \sum_j \omega(e_j) = 1$$



V-rep: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

H-rep: $\omega_{e_1} \geq 0,\ \omega_{e_2} \geq 0,$

$\omega_{e_1} + \omega_{e_2} = 1$



V-rep: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

H-rep: $\omega_{e_1} \geq 0,\ \omega_{e_2} \geq 0,\ \omega_{e_3} \geq 0,$

$\omega_{e_1} + \omega_{e_2} + \omega_{e_3} = 1$

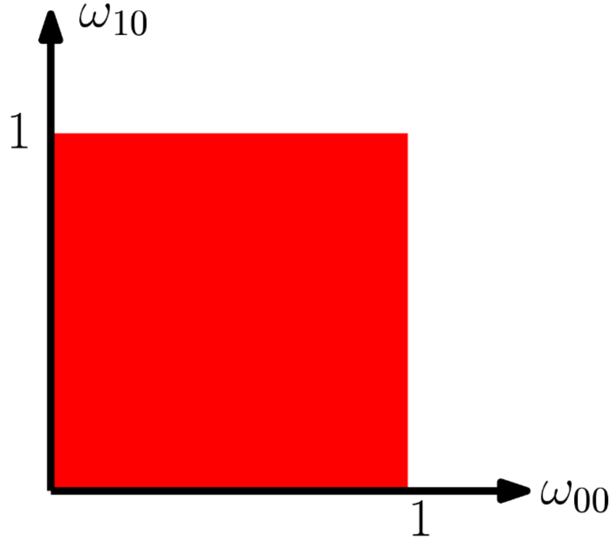
Examples

- For a gbit/squit, the state space is a square

$$\omega_{00} \geq 0, \quad \omega_{01} \geq 0, \quad \omega_{10} \geq 0, \quad \omega_{11} \geq 0$$
$$\omega_{00} + \omega_{01} = 1, \qquad \omega_{10} + \omega_{11} = 1$$

- We can use the two equalities to eliminate ω_{01} and ω_{11}. Then, we are left with

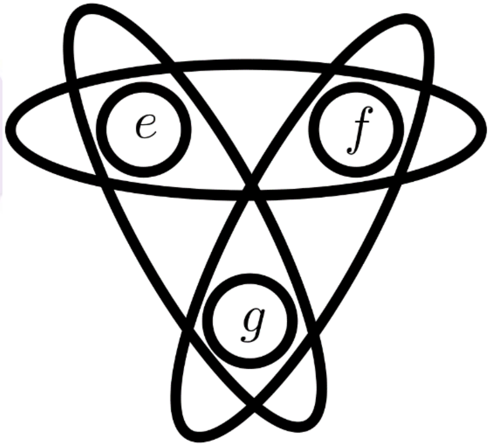
$$0 \leq \omega_{00} \leq 1, \qquad 0 \leq \omega_{10} \leq 1$$



V-rep: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

H-rep: $0 \leq \omega_{00} \leq 1, \; 0 \leq \omega_{10} \leq 1$

Examples



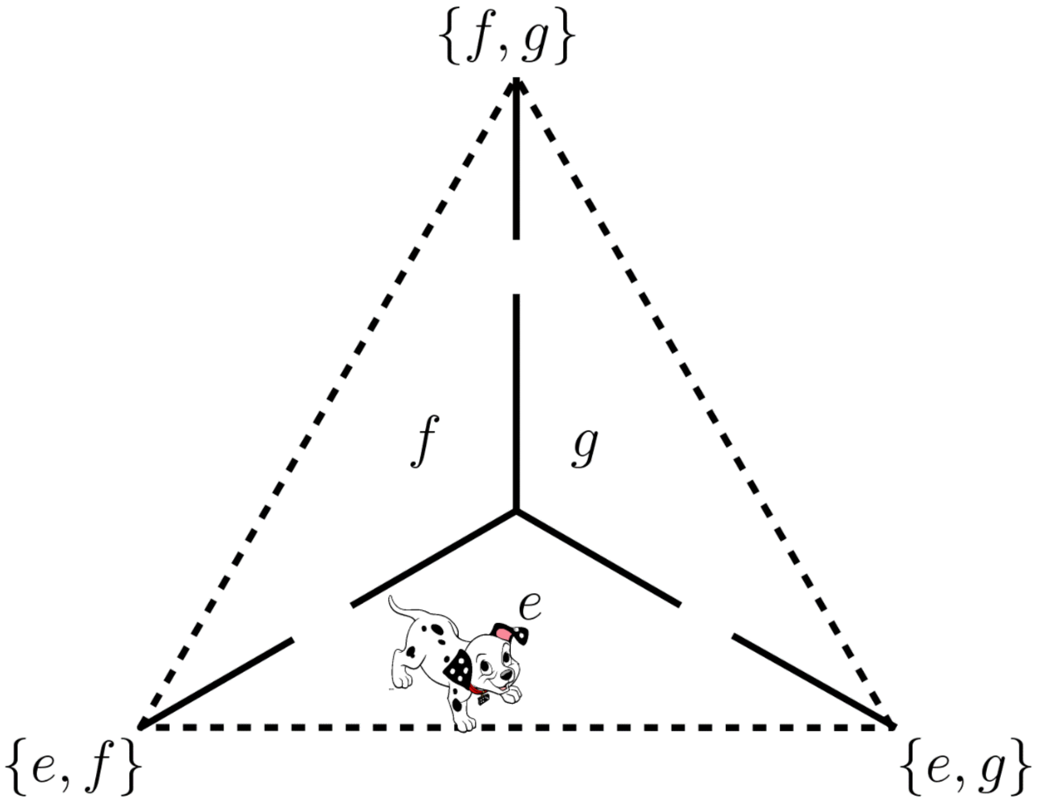
- The Specker triangle only has one state ☹
- Let's look at the equalities:

$$\omega_e + \omega_f = 1, \qquad \omega_f + \omega_g = 1, \qquad \omega_g + \omega_e = 1$$

- The first one gives us $\omega_f = 1 - \omega_e$. Substituting into the second gives $\omega_g - \omega_e = 0$.
- We now have two simultaneous equations in two variables, with solution $\omega_e = \omega_g = \frac{1}{2}$.
- Hence, also $\omega_f = \frac{1}{2}$.
- Note, one can also find test spaces with no states 😭

The Friendly Puppy

- How can we make sense of this weird state?



A puppy is in a transparent triangular cage, partitioned into 3 compartments by opaque walls with a door in them. The light is switched off. You stand at one of the corners of the cage and switch the light on. You can only see two compartments and you observe which one the puppy is in.

You always see the puppy, with 50/50 probability in one of the two compartments you can see.

Reason: The puppy is friendly and moves to one of the compartments nearest you while the light is still off.

Measurement Contextuality

- Measurement contextuality occurs when the way you make a measurement affects what happens.
  - You are not simply observing something that is independent of your choice of measurement.
- This is a feature of quantum theory that we will explore later. What is puzzling about it?
  - If the puppy story were really true, then there is no real reason why $\text{Prob}(e)$ should be independent of whether we measure $\{e, f\}$ or $\{g, e\}$. For example the puppy might prefer to stand on your left, so $\text{Prob}(e) = 0$ for $\{e, f\}$ and 1 for $\{e, g\}$.
  - We always observe $\omega(e) = \frac{1}{2}$ because this is the only state, but our model does not explain why.
  - This is an example of a *fine tuning*, IMO the biggest problem with quantum theory.

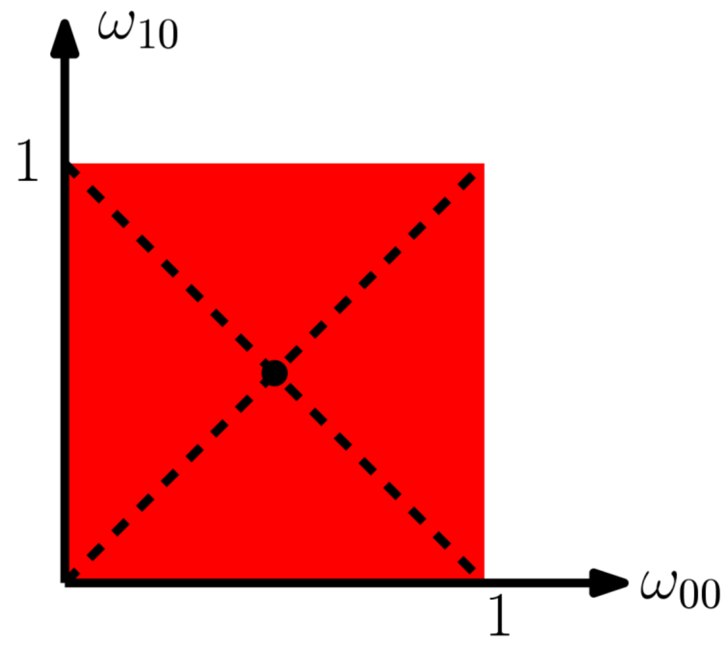
Pure state decompositions

- A state is called *pure* if it is an extreme point of Ω, otherwise it is called *mixed*.
- A mixed state can be written as a nontrivial convex combination of extremal states

$$\omega = \sum_{\mu \in \mathrm{Ext}(\Omega)} p_\mu \mu, \quad p_\mu \geq 0, \quad \sum_{\mu \in \mathrm{Ext}(\Omega)} p_\mu = 1$$

- We can always construct a preparation device that prepares ω if we have devices to prepare the pure states and classical randomness.
- The decomposition is unique iff Ω is a simplex, i.e. only for classical theories. For any other state space, there is more than one decomposition of mixed states into pure states.
  - Another feature we will see in quantum theory.

Example



V-rep: $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$

H-rep: $0 \leq \omega_{00} \leq 1,\ 0 \leq \omega_{10} \leq 1$

$$\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

3.iv) Effects

- We know that the tests of a test space represent possible measurements, but are these the only possible measurements?
  - No. We could, for example, use classical randomness to decide which test to perform and also use classical randomness to decide how to relabel the outcomes.
  - On some test spaces, there are even additional measurements that cannot be represented this way. We'll study this in the case of quantum mechanics later.

3.iv) Effects

- Let Ω be the state space of a test space. An *effect* is an affine functional $f: \Omega \to [0,1]$

$$f(p\omega + (1-p)\mu) = pf(\omega) + (1-p)f(\mu)$$
$$\forall \omega \in \Omega, \qquad 0 \leq f(\omega) \leq 1$$

- An effect is the most general way of assigning a probability to all states.
  - We want effects to be affine because one way of preparing $p\omega + (1-p)\mu$ is to have devices that prepare ω and μ and decide which one to use with probabilities p and $1-p$.
- An affine functional is an example of an affine map, where the output space is just $\mathbb{R}$. As we know, this is not necessarily a linear function on the state space.

The State/Effect Cones

- But we know how to deal with that. We can form a cone $C(\Omega)$ by lifting to one dimension higher and closing under positive linear combinations. This is called the *state cone*.
- All $\mu \in C(\Omega)$ can be written as $\mu = \alpha\omega$ for some $\omega \in \Omega$.
- In $\mathbb{R}^X$, the state cone is specified by

$$\forall e \in X, \qquad \omega(e) \geq 0$$

$$\forall E \in \Sigma, \ \sum_{e \in E} \omega(e) = \alpha \text{ for some } \alpha \geq 0.$$

- This represents the space of unnormalized states – divide by α to get a normalized state.
- On this space, an effect is represented by a *linear* functional.
- Since we require $f(\omega) \geq 0$, effects lie inside the dual cone $C^*(\Omega)$, so we call this the *effect cone*.
- However, not every element of the dual cone is an effect because we also have to satisfy $f(\omega) \leq 1$ for $\omega \in \omega$.

The Space of Effects

- Therefore the *space of effects* $\mathcal{E}(\Omega)$ is the set of all $f \in C^*(\Omega)$ such that $f(\omega) \leq 1$ for all $\omega \in \Omega$.
  - Note: We can check this condition just for the extreme points of Ω.
- We can express this a different way. Let $u \in \ (\Omega)$ be the *unit effect*

$$u(\omega) = 1 \text{ for all } \omega \in \Omega$$

- and let 0 be the zero effect $0(\omega) = 0$.
- Define a partial order relation $\preccurlyeq$ on $C^*(\Omega)$ by

$$f \preccurlyeq g \quad \Leftrightarrow \quad f(\omega) \leq g(\omega) \text{ for all } \omega \in C(\Omega).$$

- Then, $\mathcal{E}(\Omega)$ is the set of all linear functionals f such that

$$0 \preccurlyeq f \preccurlyeq u.$$

Outcomes as effects

- An outcome $e \in X$ obviously defines an effect via
$$e(\omega) = \omega(e)$$
  - I am abusing notation by using the same symbol for the outcome and the effect.
- In $\mathbb{R}^X$ these effects are represented by the standard basis vectors because
$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \omega_a \\ \omega_b \\ \vdots \\ \omega_e \\ \vdots \\ \omega_z \end{pmatrix} = \omega(e)$$

The Unit Effect

- Since $\sum_{e \in E} \omega(e) = 1$ for any test E and any state ω, we can represent the unit effect in $\mathbb{R}^X$ by picking a test and setting

$$u_e = \begin{cases} 1, e \in E \\ 0, e \notin E \end{cases}$$

- The representation of an effect in $\mathbb{R}^X$ is generally non-unique because $C(\Omega)$ has a lower dimension than $\mathbb{R}^X$ due to the equalities defining Ω. If we instead use a representations of the same dimension as $C(\Omega)$ by eliminating equalities then the effects will be represented by unique vectors.