Quantum Foundations Lecture 2

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HSC112

2) Mathematical Background

i. Vector Spaces
ii. Convex Cones
iii. Convex Sets
iv. Convex Polytopes
v. Inner Products
vi. Dual Spaces

2.i) Vector Spaces

- A **vector space** consists of:
  - A set of **vectors** $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \cdots$.
  - A set of **scalars** $a, b, c, \cdots$.
    - In this section, the scalars will usually be the real numbers. Later, quantum theory makes heavy use of complex vector spaces.
  - A rule of the addition of vectors: $\boldsymbol{x} + \boldsymbol{y}$
  - A rule for multiplying a vector by a scalar: $a\boldsymbol{x}$

Vector addition rules

- The addition rule must satisfy:
  - If $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors, then $\boldsymbol{x}+\boldsymbol{y}$ is a vector.
  - Commutativity: $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$.
  - Associativity: $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}=\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$.
  - There exists a vector $\mathbf{0}$ such that, for all vectors $\boldsymbol{x}$
    $$\mathbf{0}+\boldsymbol{x}=\boldsymbol{x}+\mathbf{0}=\boldsymbol{x}.$$
  - For each vector $\boldsymbol{x}$, there exists a unique vector $-\boldsymbol{x}$, such that
    $$\boldsymbol{x}+(-\boldsymbol{x})=(-\boldsymbol{x})+\boldsymbol{x}=\mathbf{0}.$$

Scalar Multiplication Rules

- The scalar multiplication rule must satisfy:
  - For every scalar a and every vector $\boldsymbol{x}$, $a\boldsymbol{x}$ is a vector.

    This implies $a\boldsymbol{x} + b\boldsymbol{y}$ is always a vector for any scalars a, b and any vectors $\boldsymbol{x}, \boldsymbol{y}$.
  - Distributivity:

    $$a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y} \quad \text{and} \quad (a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}.$$
  - Associativity: $a(b\boldsymbol{x}) = (ab)\boldsymbol{x}$.
  - There exists a unit scalar 1 and a zero scalar 0 such that, for all vectors $\boldsymbol{x}$,

    $$1\boldsymbol{x} = \boldsymbol{x} \quad \text{and} \quad 0\boldsymbol{x} = \boldsymbol{0}.$$

Examples of vector spaces

- $\mathbb{R}^n$: The set of n-dimensional column vectors with real components.
- $\mathbb{C}^n$: The set of n-dimensional column vectors with complex components.

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$$

- Addition is component-wise addition. Scalars are the real numbers for $\mathbb{R}^n$ and complex numbers for $\mathbb{C}^n$.

Examples of vector spaces

- $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$: The set of infinite dimensional real or complex column vectors.
- The set of real or complex-valued functions of a real number x.
  - Addition is $(f+g)(x) = f(x) + g(x)$.
  - Scalars are real or complex numbers with

$$(af)(x) = af(x)$$

Dimension and bases

- A set of N nonzero vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ is *linearly independent* iff the only solution to the equation

$$\sum_{n=1}^{N} a_n \boldsymbol{x}_n = 0$$

is $a_1 = a_2 = \cdots = a_n = 0$.

- Otherwise the vectors are *linearly dependent*, and one of the vectors can be written as a sum of the others:

$$\boldsymbol{x}_j = \sum_{n=1}^{j-1} b_n \boldsymbol{x}_n + \sum_{n=j+1}^{N} b_n \boldsymbol{x}_n \qquad \text{with} \qquad b_n = -\frac{a_n}{a_j}$$

Dimension and bases

- The *dimension* d of a vector space is the maximum number of linearly independent vectors.
- A *basis* $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_d$ for a vector space is a linearly independent set of maximum size.
  - All vectors can be written as:

$$\boldsymbol{y} = \sum_{n=1}^{d} b_n \boldsymbol{x}_n$$

for some components b_n.

Examples

- For $\mathbb{R}^n$ and $\mathbb{C}^n$, the vectors

$$\boldsymbol{e}_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

j^{th} component

are a basis, but not the only one. E.g. for $\mathbb{R}^2$ and $\mathbb{C}^2$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is also a basis.

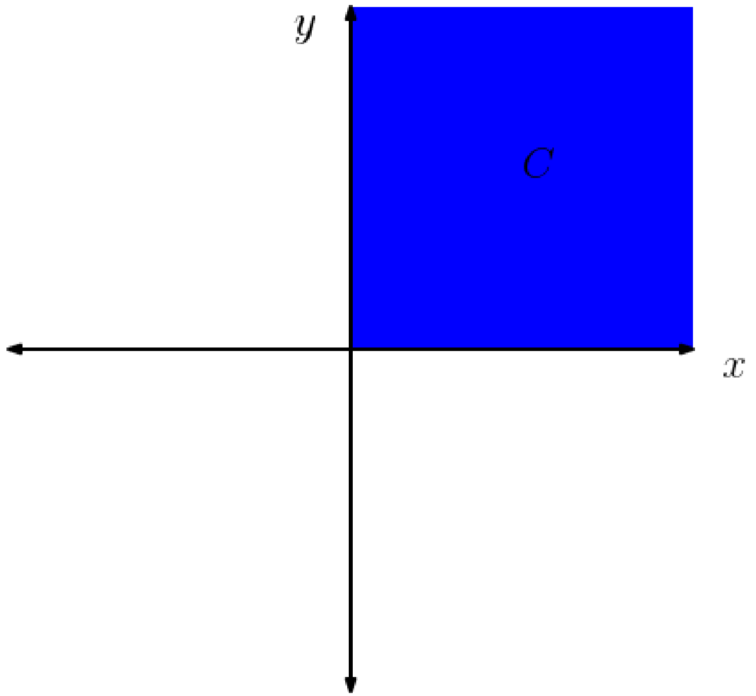
Examples

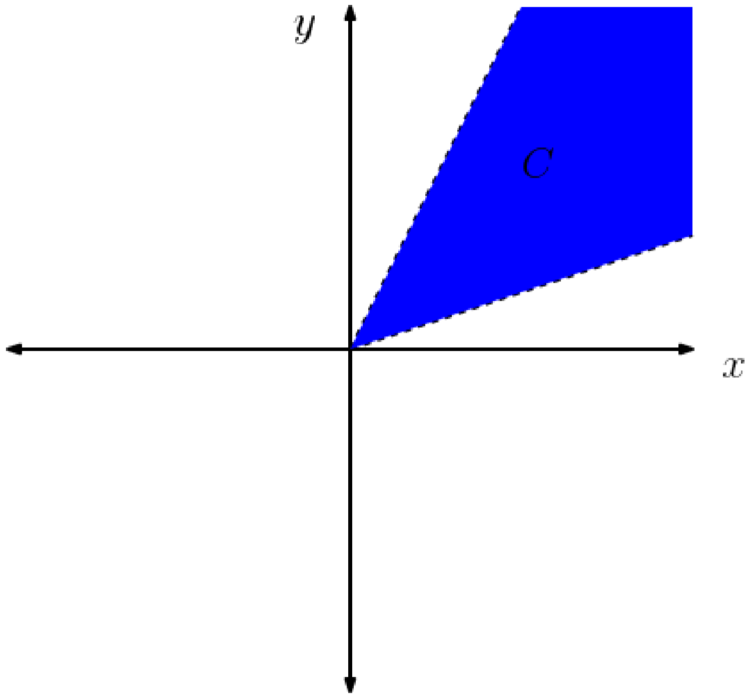
- $\mathbb{R}^n$ and $\mathbb{C}^n$ have dimension n.
- $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$ have dimension (countable) infinity.
- The space of functions of a real variable has dimension (uncountable) infinity.

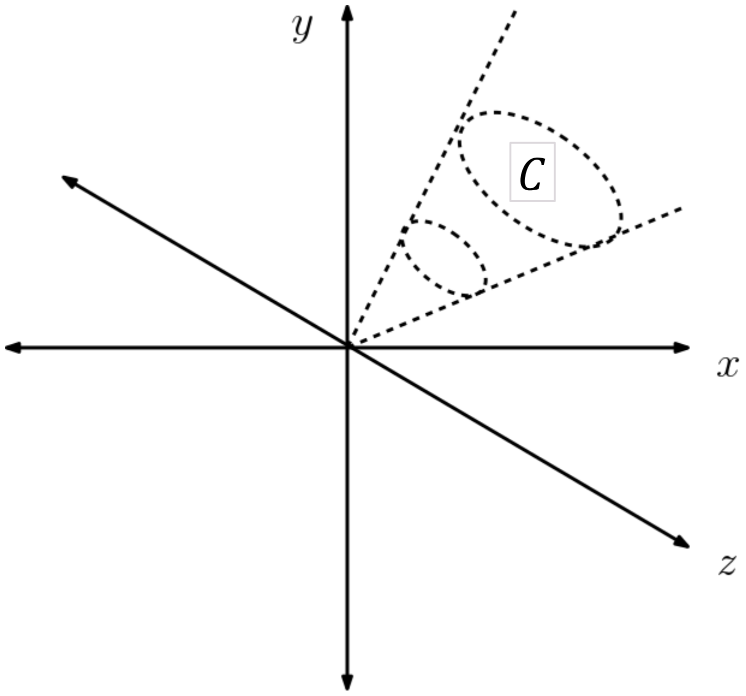
2.ii) Convex Cones

- A *convex cone* C is a subset of a real vector space such that, if $\boldsymbol{x}, \boldsymbol{y} \in C$ then $\alpha\boldsymbol{x} + \beta\boldsymbol{y} \in C$ for any positive (or zero) scalars $\alpha, \beta \geq 0$.
- Note: A convex cone always includes the origin.
- A convex cone is *salient* if, for every $\boldsymbol{x} \in C$, $-\boldsymbol{x} \notin C$.
  - We will always work with salient convex cones, so when I say "cone", that is what I mean.
- Example: The set of vectors with positive components.

Examples







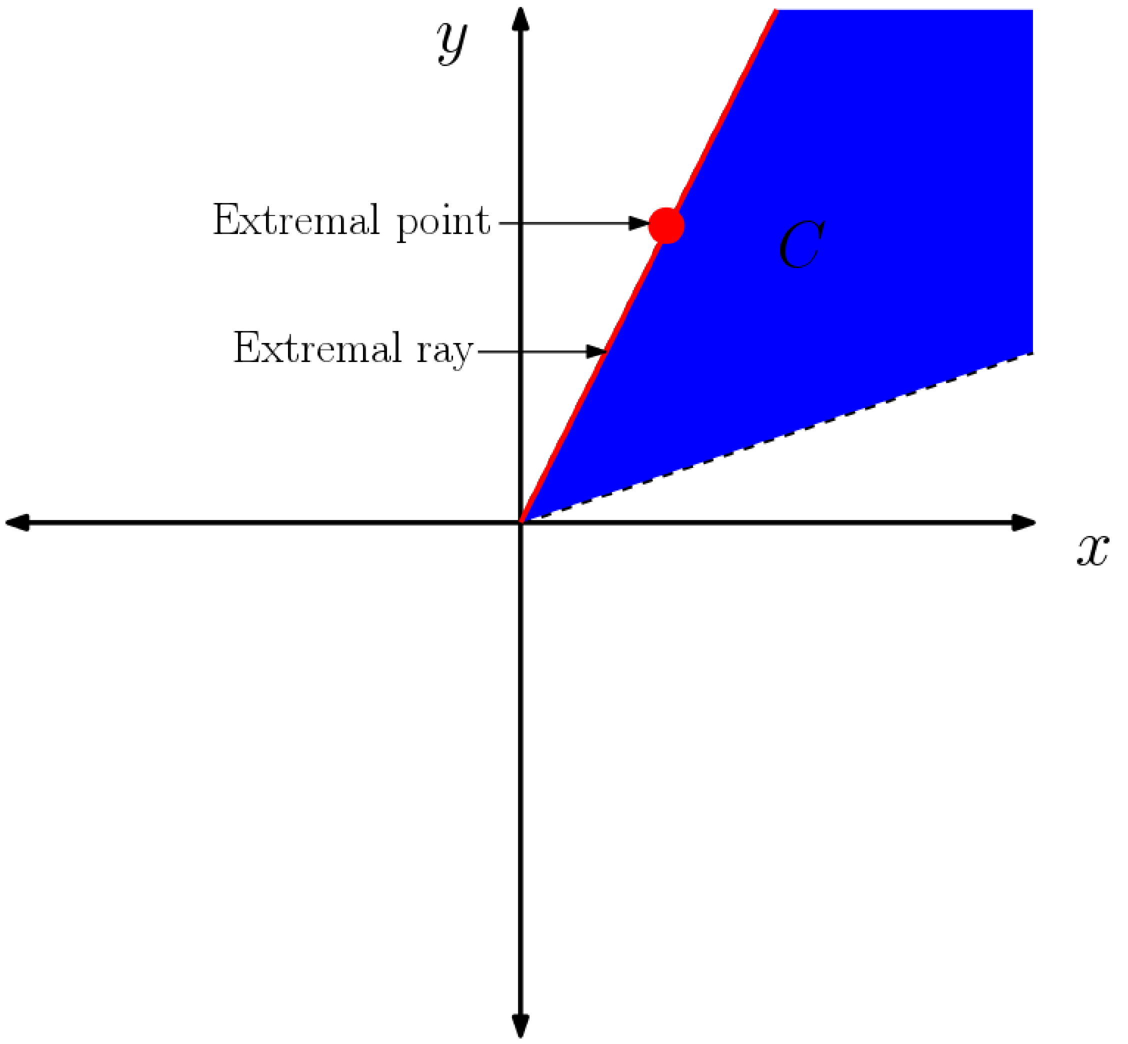
Extremal Rays

- A vector $\boldsymbol{x}$ is called an *extreme point* of C if, whenever

$$\boldsymbol{x} = \alpha \mathbf{y} + \beta \boldsymbol{z} \text{ for } \mathbf{y}, \boldsymbol{z} \in C \text{ and } \alpha, \beta \geq 0,$$

then $\boldsymbol{y} = \gamma \boldsymbol{x}$ and $\boldsymbol{z} = \delta \boldsymbol{x}$ for $\gamma, \delta \geq 0$.

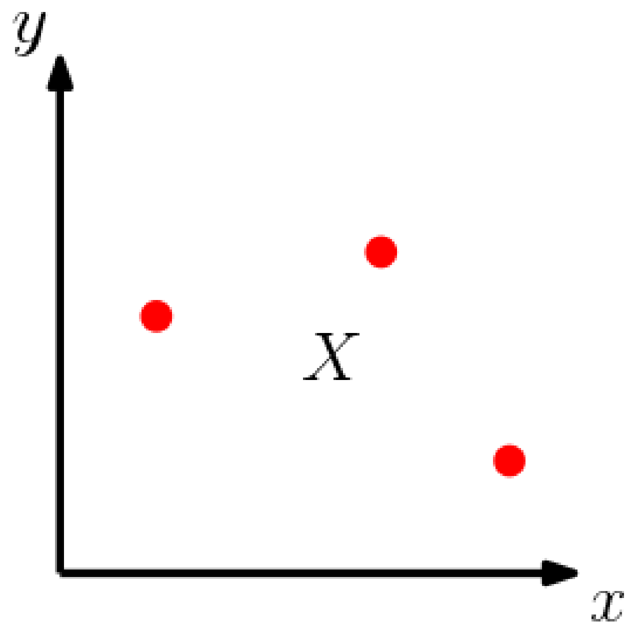
- An *extremal ray* of a cone C is the set of points $\alpha \boldsymbol{x}$ for $\alpha \geq 0$ where $\boldsymbol{x}$ is an extreme point.

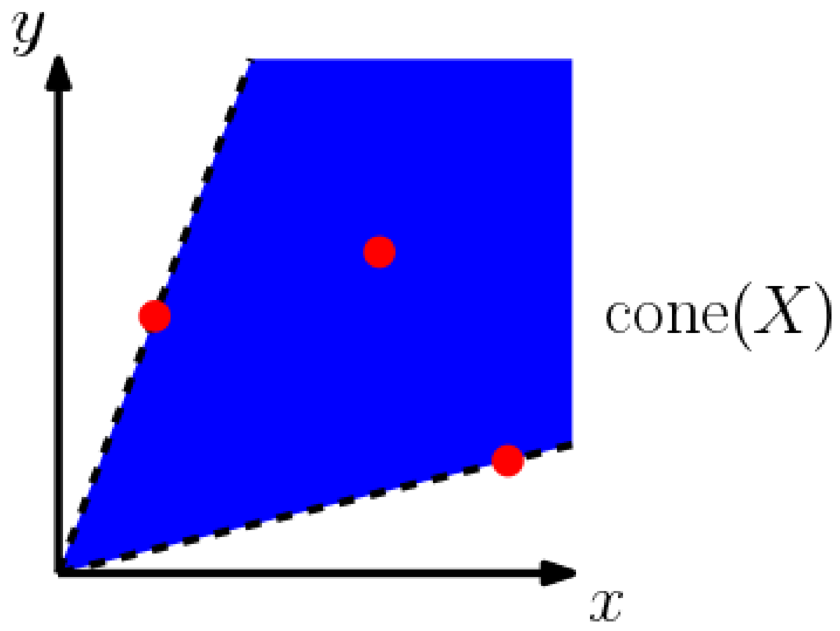


Conic Hulls

- Given a set X of vectors in $\mathbb{R}^n$, the *conic hull* of X is a convex cone, denoted $\mathbf{cone}(X)$, consisting of the set of all points of the form

$$\sum_j \alpha_j \boldsymbol{x}_j \text{ for } \alpha_j \geq 0 \text{ and } \boldsymbol{x}_j \in X$$





Carathéodory's Theorem for Cones

- **Theorem:** Let X be a set of vectors in $\mathbb{R}^n$. Every $\boldsymbol{x} \in \mathrm{cone}(X)$ can be written as a positive combination of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m \in X$ that is linearly independent, so, in particular, $m \leq n$.
- Proof: Let $\boldsymbol{x} \in \mathrm{cone}(X)$ and let m be the smallest integer such that $\boldsymbol{x} = \sum_{j=1}^{m} \alpha_j \boldsymbol{x}_j$ for $\alpha_j \geq 0$ and $\boldsymbol{x}_j \in X$.
- If the vectors were linearly dependent, there would exist $a_1, a_2 \dots, a_m$ with at least one a_j positive such that

$$\sum_{j=1}^{m} a_j \boldsymbol{x}_j = 0.$$

Carathéodory's Theorem for Cones

- Let β be the largest positive number such that
$$\gamma_j = \alpha_j - \beta a_j \geq 0 \text{ for all } j.$$
- At least one of the γ_j's is zero and
$$\boldsymbol{x} = \sum_{j=1}^{m} \gamma_j \boldsymbol{x}_j$$
- This has $m - 1$ nonzero terms, so contradicts the assumption that m was the smallest possible integer.

Krein-Milman theorem for cones

- **Theorem**: Every convex cone C in $\mathbb{R}^n$ is the conic hull of its extreme points.
- Let $\mathrm{Ext}(C)$ be the set of extreme points of C. Then, the theorem says
$$C = \mathrm{cone}(\mathrm{Ext}(C))$$
- **Corollary:** Every $\boldsymbol{x} \in C$ can be written as a positive combination of at most n extreme points.

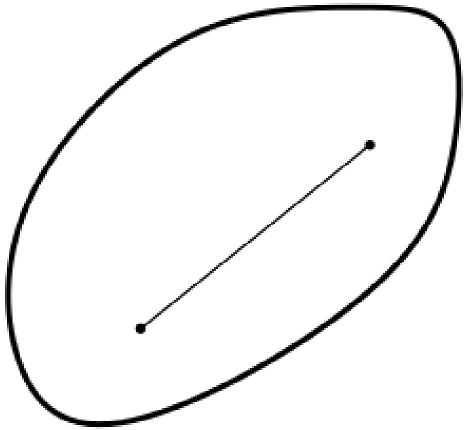
2.iii) Convex Sets

- A convex set S in $\mathbb{R}^n$ is a set of vectors such that, whenever $\boldsymbol{x}, \boldsymbol{y} \in S$ then

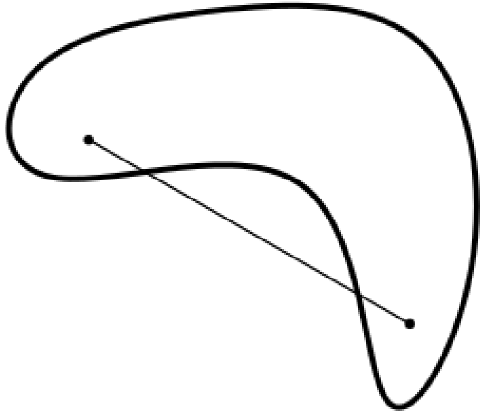
$$\alpha\boldsymbol{x} + \beta\boldsymbol{y} \in S \quad \text{for all } \alpha, \beta \geq 0,\ \alpha + \beta = 1.$$

- Note that we will always be interested in *bounded*, *closed* convex sets.
  - *Bounded* means that all the components of $\boldsymbol{x}$ satisfy $|x_j| < N$ for some positive, but arbitrarily large N.
  - *Closed* means that, for any convergent sequence of vectors in S, the limit point is also in S.
- All closed convex sets are bounded. We will see that *salient* cones are related to bounded convex sets.

Examples



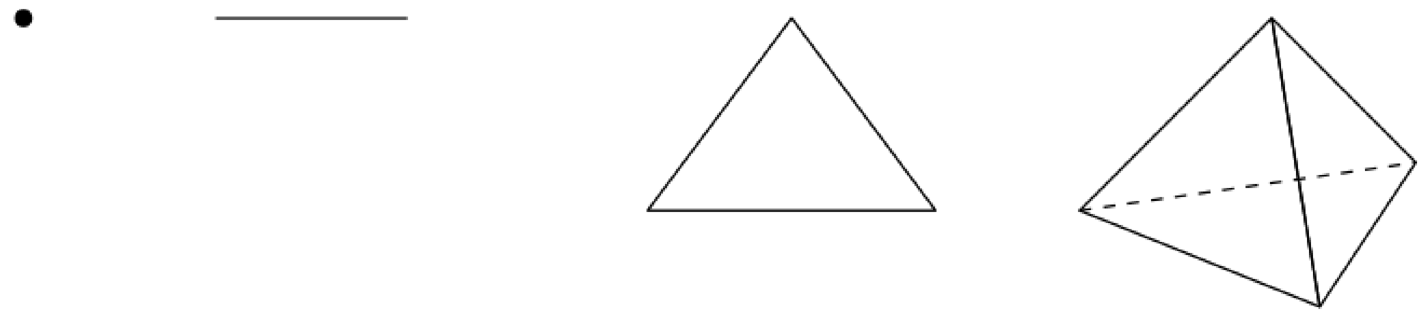
Convex: Line connecting any two points is included in S.



Not Convex: There is a line connecting two points that is not in S.

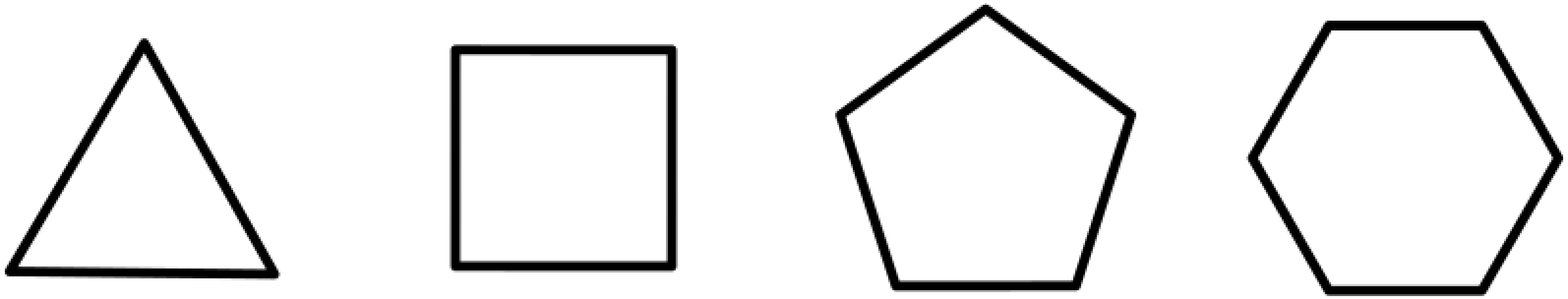
Simplexes:

point line triangle tetrahedron

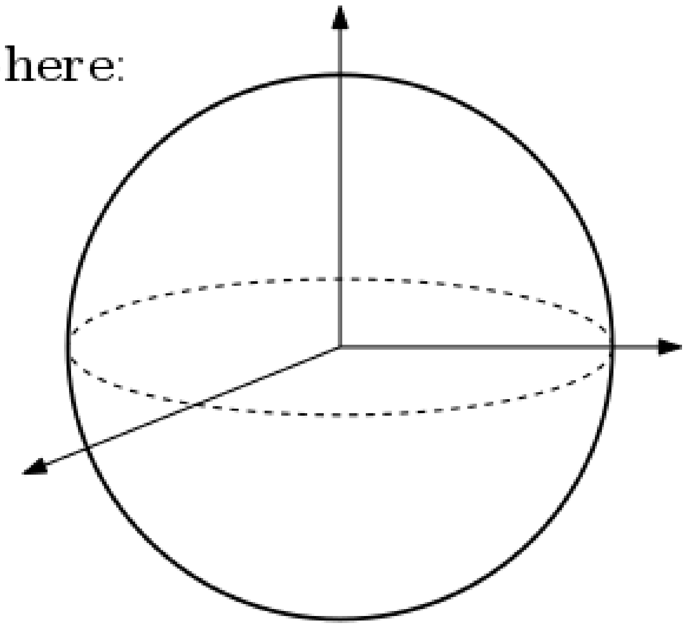


Examples

Polygons:



Sphere:



Convex Combinations

- If we can write $\boldsymbol{y} = \sum_{j=1}^{n} \alpha_j \boldsymbol{x}_j$ for $\alpha_j \geq 0$ and $\sum_j \alpha_j = 1$ then $\boldsymbol{y}$ is called a *convex combination* of the $\boldsymbol{x}_j$'s.
- **Proposition**: For a convex set S, any convex combination of vectors in S is in S.
- Proof: Consider $\boldsymbol{y} = \sum_{j=1}^{n} \alpha_j \boldsymbol{x}_j$ where $\boldsymbol{x}_j \in S$.
- Define $\alpha_2' = \alpha_1 + \alpha_2$ and $\boldsymbol{x}_2' = \frac{\alpha_1}{\alpha_2'} \boldsymbol{x}_1 + \frac{\alpha_2}{\alpha_2'} \boldsymbol{x}_2$.
- $\boldsymbol{x}_2'$ is in S by definition and $\alpha_2' + \sum_{j=3}^{n} \alpha_j = 1$, so

$$\boldsymbol{y} = \alpha_2' \boldsymbol{x}_2' + \sum_{j=3}^{n} \alpha_j \boldsymbol{x}_j$$

  is a convex combination of $n - 1$ terms in S.
- Proceeding by induction, we can reduce this to two terms, which is in S by definition.

Extreme Points

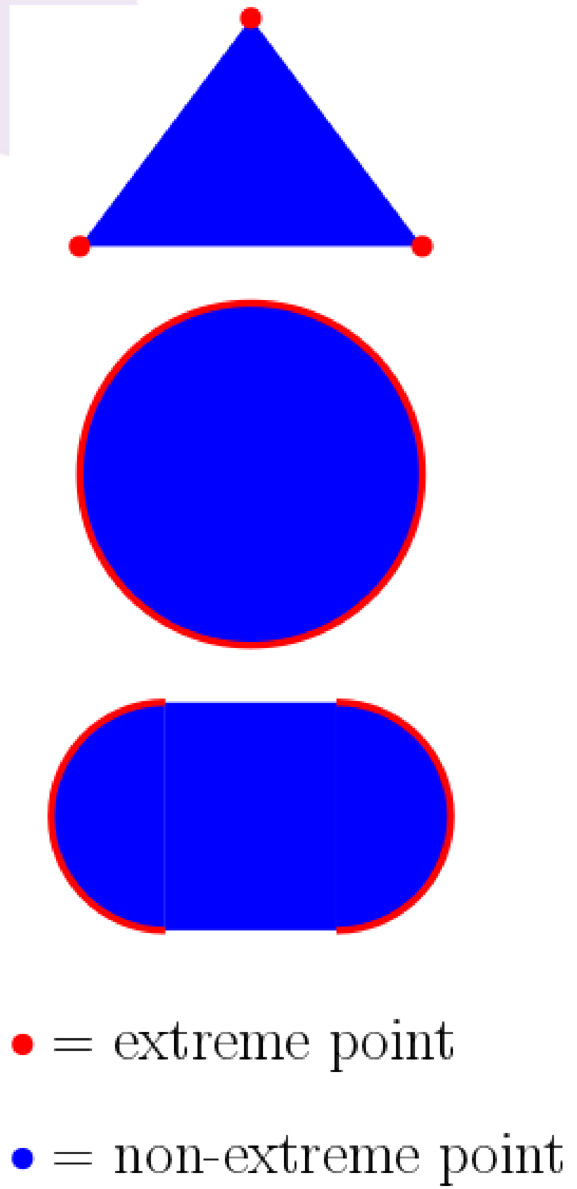
- A vector $\boldsymbol{x}$ is called an *extreme point* of S if whenever

$$\boldsymbol{x} = \alpha \boldsymbol{y} + \beta \boldsymbol{z}$$

for $\boldsymbol{y}, \boldsymbol{z} \in S, \alpha, \beta \geq 0, \alpha + \beta = 1,$

then $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{x}.$

- Extreme points always lie on the boundary of S, but boundary points are not necessarily extreme.

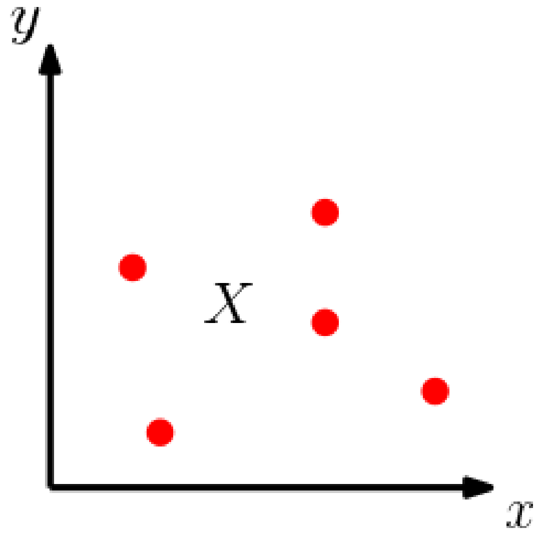


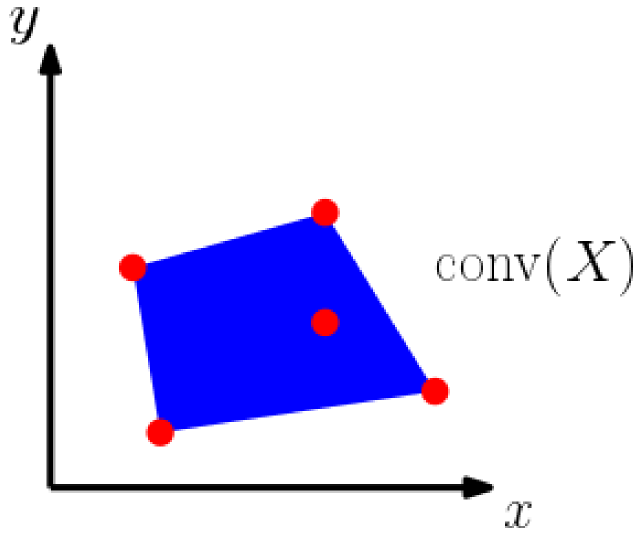
Convex Hulls

- Given a set X of vectors in $\mathbb{R}^n$, the *convex hull* of X, denoted $\text{conv}(X)$ is the set of all vectors of the form

$$\boldsymbol{y} = \sum_j \alpha_j \boldsymbol{x}_j$$

where $\boldsymbol{x}_j \in X$, $\alpha_j \geq 0$, and $\sum_j \alpha_j = 1$.





Converting a Convex Set Into a Convex Cone

- A convex set of dimension d can be *lifted* to form a convex cone of dimension $d+1$.
- There are many ways to do this. We will define the *standard lifting* as follows:
- For every vector $\boldsymbol{x}$ in the convex set $S \subseteq \mathbb{R}^d$, define the vector

$$\boldsymbol{x}' = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \in \mathbb{R}^{d+1}$$

- Let X be the set of such vectors and define the cone $C = \text{cone}(X)$.

Examples

- The convex set S is embedded as a cross-section of the cone C.
- If $\boldsymbol{x}$ is an extreme point of S then $\alpha\boldsymbol{x}'$ is an extremal ray of C, i.e.

$$\begin{pmatrix} \alpha \\ \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_d \end{pmatrix} \text{ is extremal.}$$