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Plausibility Measures on Test Spaces

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Introduction

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Weaker predictive structures

- We normally say that, at a minimum, a physical theory should supply *probabilities* for the outcomes of any experiment.
- But it is possible to contemplate weaker predictive structures, e.g.
  - *Possibilistic*/modal/relational theories: For any event A we can say whether A is possible or impossible, e.g. supports on test spaces[1].
  - *Comparative* theories: For events A and B, it may be possible to say that A is less likely than B, without giving precise numerical probabilities, and relative likelihood may only be a partial order.
- *Plausibility measures*[2], unify probabilistic, comparative, and possibilistic predictions. They have only been developed for classical theories. We generalize to *test spaces*.

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[1] D. Foulis et. al., *Found. Phys.* 13:813–842 (1983). C. Randall and D. Foulis, *Found. Phys.* 13:843–857 (1983). D. Foulis et. al., *IJTP* 31:789–807 (1992).

[2] N. Fiedman and J. Halpern, *Proc. 11th Conference on Uncertainty in Artificial Intelligence (UAI1995)* (1995). arXiv:1302.4947.

- Adversarial scenarios:
  - Consider a device with n settings that prepares ρ_j when the setting is j.
  - You want to bet on the outcomes of a quantum experiment described by a POVM $\{E_k\}$. However, the bookmaker gets to choose the setting *after* you have placed your bets.
  - It does not make sense to assign a prior probability to the setting because it is chosen adversarialy.
  - However, it is still safe to say that E_k is less likely than E_m if $\text{Tr}\left(E_k\rho_j\right) < \text{Tr}\left(E_m\rho_j\right)$ for all j.

Motivation: Generalizing quantum theory

- Requiring probabilities restricts the possible generalizations of quantum theory.
  - E.g. Cannot have quantum theory with $\mathbb{C}$ replaced by a finite field because vector spaces over finite fields have no inner product.
  - Schumacher and Westmoreland constructed a *possibilistic* quantum theory over finite fields[3].
  - More generally, some well-defined operational structures, e.g. test spaces, quantum logics, contextuality scenarios etc. have no probabilistic states, but they do have possibilistic and comparative states.

[3] B. Schumacher and M. Westmoreland, *Proc. 7th International QPL Workshop* (2010). arXiv:1010.2929

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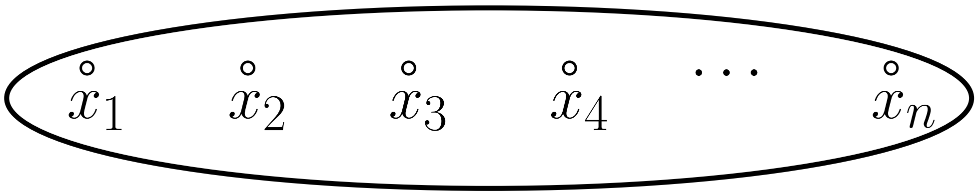
Test spaces

- Definition: A *test space* (X, Σ) consits of
  - A set X of *outcomes*.
  - A set Σ of subsets of X such that

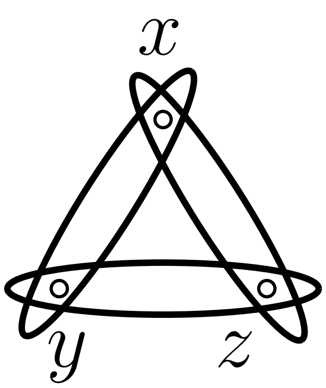
$$\bigcup_{T \in \Sigma} T = X.$$

- A set $T \in \Sigma$ is called a *test*.
- A test space is called *finite* if X is finite (in which case the test space is a *hypergraph*).
- It is *locally finite* if each $T \in \Sigma$ is finite.

- Classical test space: $(\{x_1, x_2, \ldots, x_n\}, \{\{x_1, x_2, \ldots, x_n\}\})$



- Specker's triangle: $(\{x, y, z\}, \{\{x, y\}, \{y, z\}, \{z, x\}\})$



- Quantum test space: $(P(\mathcal{H}), b(\mathcal{H}))$, where
  - $P(\mathcal{H})$ = the set of unit vectors in $\mathcal{H}$ (up to global phases).
  - $b(\mathcal{H})$ = the set of orthonormal bases (up to global phases).

- Definition: An *event* on a test space (X, Σ) is a subset of a test. $E(X, \Sigma)$ denotes the set of events.
- Examples:
  - Classical: An event is any subset of $X = \{x_1, \ldots, x_n\}$.

$$E(X, \{X\}) = 2^X.$$

  - Specker:

$$E(\{x, y, z\}, \{\{x, y\}, \{y, z\}, \{z, x\}\}) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{z, x\}\}.$$

  - Quantum: An event is a subset of the vectors in an orthonormal basis. Each event can be associated with the projector onto their span.

- Definition: A *probability measure* on a (locally finite) test space is a function $\mu : E(X, \Sigma) \to [0, 1]$ such that,
  - For any test $T \in \Sigma$, $\quad \mu(T) = 1$.
  - $\mu(\emptyset) = 0$.
  - If $A, B \in E(X, \Sigma)$ are disjoint and there exists a test T such that $A \subseteq T$, $B \subseteq T$, then
  $$\mu(A \cup B) = \mu(A) + \mu(B).$$
- One implication of this is that, if $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

Plausibility measures

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- A *plausibility measure* on a test space (X, Σ) is a function Pl : $E(X, \Sigma) \to D$, where
  - $(D, \preceq)$ is a bounded poset with minimal element 0 and maximal element 1.
  - For any test $T \in \Sigma$, $\quad$ Pl$(T) = 1$.
  - Pl$(\emptyset) = 0$.
  - If $A, B \in E(X, \Sigma)$ satisfy $A \subseteq B$, then Pl$(A) \preceq$ Pl(B).

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Examples

- A probability measure is a plausibility measure.
- Let $D = \{0, 1\}$ with $0 \prec 1$. A plausibility measure Pl such that, for every test $T \in \Sigma$ there exists an $x \in T$ with $\text{Pl}(x) = 1$ is called a *possibility measure*.
- Given a set $\{\mu_j\}_{j=1}^n$ of probability measures, let

$$\text{Pl}(A) = (\mu_1(A), \mu_2(A), \ldots, \mu_n(A))$$

and define the poset:

- $D := [0, 1]^{\times n}$
- Ordering: $(a_1, a_2, \ldots, a_n) \preceq (b_1, b_2, \ldots, b_n)$ if $a_j \leq b_j$ for all j
- Minimal element: $0 = (0, 0, \ldots, 0)$
- Maximal element $(1, 1, \ldots, 1)$

Then, we have a plausibility measure with $\text{Pl}(A) \preceq \text{Pl}(B)$ iff $\mu_j(A) \leq \mu_j(B)$ for all j.

Agreement

- Definition: A plausibility measure Pl on a test space (x, Σ) *agrees* with a probability measure μ if

$$\text{Pl}(A) \preceq \text{Pl}(B) \qquad \Leftrightarrow \qquad \mu(A) \leq \mu(B).$$

It *almost agrees* with μ if

$$\text{Pl}(A) \preceq \text{Pl}(B) \qquad \Rightarrow \qquad \mu(A) \leq \mu(B).$$

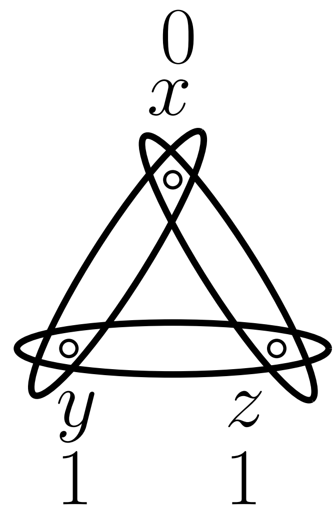
- Agreement implies that the image of Pl is totally ordered and that

$$\mu(A) = \mu(B) \qquad \Rightarrow \qquad \text{Pl}(A) = \text{Pl}(B).$$

- Almost agreement $+$ these two additional conditions is the same as agreement. In general it is weaker.

Disagreeable plausibility measures

- Not all plausibility measures agree with a probability measure.



- Must have

$$\mu(x) + \mu(y) = 1 \qquad \mu(y) + \mu(z) = 1$$

but these assignments imply $\mu(x) + \mu(y) = 0 + 1 - \mu(z) < 1$.

- There are examples for classical test spaces as well.

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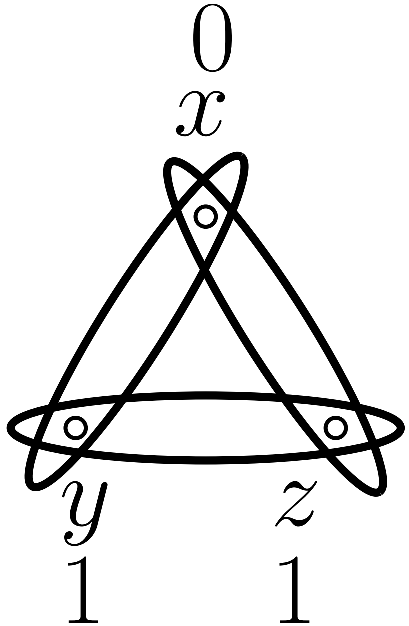
The Archimedean condition

The Archimedean condition

- Needed: An order theoretic counterpart of additivity.
- Two list of events, $(A_1, A_2, \ldots, A_n)$ and $(B_1, B_2, \ldots, B_n)$ are *equivalent* if every outcome occurs the same number of times in both.
  - Example: $(\{y, z\}, \{x, z\}, \{x\})$ and $(\{x, y\}, \{x, z\}, \{z\})$.
- Definition: A plausibility measure is *Archimedean* if, whenever $(A_1, A_2, \ldots, A_n)$ and $(B_1, B_2, \ldots, B_n)$ are equivalent and

$$\mathrm{Pl}(A_1) \preceq \mathrm{Pl}(B_1), \quad \ldots, \quad \mathrm{Pl}(A_{n-1}) \preceq \mathrm{Pl}(B_{n-1}),$$

then $\mathrm{Pl}(A_n) \succeq \mathrm{Pl}(B_n)$.



- Consider $(\{y,z\},\{x\})$ and $(\{x,y\},\{z\})$. We have,

$$\text{Pl}(\{y,z\}) \preceq \text{Pl}(\{x,y\}),$$

but $\text{Pl}(x) \prec \text{Pl}(z)$.

Main results

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Classical theorems

- Theorem: A plausibility measure Pl on a finite classical test space $(X, \{X\})$ almost agrees with some probability measure μ iff it is Archimedean.

- Theorem: A plausibility measure Pl on a finite classical test space $(X, \{X\})$ agrees with some probability measure μ iff the image of Pl is totally ordered and it is Archimedean.

- Theorem: A plausibility measure Pl on a finite test space (X, Σ) almost agrees with some probability measure μ iff it is Archimedean.

- Theorem: A plausibility measure Pl on a finite test space (X, Σ) agrees with some probability measure μ iff the image of Pl is totally ordered and it is Archimedean.

Locally finite test spaces

- Theorem: A plausibility measure Pl on a locally finite test space (X, Σ) almost agrees with some probability measure μ iff it is Archimedean.

- Theorem: There exist locally finite test spaces (X, Σ) on which there are plausibility measures Pl that are Archimedean and have totally ordered image, but do not agree with any probability measure μ.

Quantum counter-example

- Let $(P(\mathbb{C}^d), b(\mathbb{C}^d))$ be a quantum test space and let $\{\rho_j\}_{j=1}^{d^2}$ be a tomographically complete set of states. Define a plausibility measure via

$$\begin{aligned} \text{Pl}(\Pi) &= \text{Pl}(\Pi') & \text{if } \text{Tr}\left(\Pi\rho_j\right) = \text{Tr}\left(\Pi'\rho_j\right) \text{ for all j} \\ \text{Pl}(\Pi) &\prec \text{Pl}(\Pi') & \text{if } \text{Tr}\left(\Pi\rho_k\right) < \text{Tr}\left(\Pi'\rho_k\right) \\ \text{Pl}(\Pi) &\succ \text{Pl}(\Pi') & \text{if } \text{Tr}\left(\Pi\rho_k\right) > \text{Tr}\left(\Pi'\rho_k\right), \end{aligned}$$

  where k is the smallest value such that $\text{Tr}\left(\Pi\rho_k\right) \neq \text{Tr}\left(\Pi'\rho_k\right)$.
- Pl is totally ordered and it can be shown to be Archimedean.
- By Gleason's theorem all probability measures on $(P(\mathbb{C}^d), b(\mathbb{C}^d))$ are quantum states.
- For every quantum state ρ, there are pairs of unit vectors $|\psi\rangle, |\phi\rangle$ that get assigned the same probability, e.g. equal superposition of two eigenvectors of ρ with a differing relative phase.
- However, by tomographic completeness of $\{\rho_j\}_{j=1}^{d^2}$, no two unit vectors are assigned the same plausibility.

The strong Archimedean condition

- Definition: A plausibility measure Pl on a test space (X, Σ) is *strongly Archimedean* if, for every $A, B \in E(X, \Sigma)$, if, for every $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ and lists of events $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_m)$ such that Pl$(A_j) \preceq$ Pl(B_j) and the two lists

$$(kA, A_1, \ldots, A_m), \qquad (kB, B_1, \ldots, B_m)$$

differ in a set of outcomes (with multiplicity) that fits into at most k/n tests, then Pl$(A) \succeq$ Pl(B).

- The same condition was used for the measure theoretic classical case to derive countable additivity[4].

[4] A. Chateauneuf and J. Jaffray, *J. Math. Psychology* 28(2), pp. 191–204 (1984).

- Theorem: A plausibility measure Pl on a locally finite test space (X, Σ) with finite dimensional state space agrees with some probability measure μ iff the image of Pl is totally ordered and it is strongly Archimedean.

Proof idea

Order unit spaces

- Let V be a vector space over an ordered subfield $\mathbb{F}$ of the real numbers (e.g. the rationals).
- Definition: A subset $C \subseteq V$ is a *convex cone if*

$$a \in C, b \in C \Rightarrow a + b \in C \qquad \lambda \in \mathbb{F}_{\geq 0}, a \in C \Rightarrow \lambda a \in C.$$

  $a \leq b$ is used to denote $b - a \in C$.

- Definition: An *order unit space* is a triple (V, C, u), where $C \subseteq V$ is a convex cone, $u \geq 0$ is a distinguished element called the *order unit* such that

  1. $-u \not\geq 0$,
  2. For any $a \in V$, there is a $\lambda \in \mathbb{F}$ such that $\lambda u + a \geq \lambda u$.

- Definition: A *probability measure* ω on (V, C, u) is an $\mathbb{F}$-linear functional $\omega : V \to \mathbb{R}$ with $\omega(a) \geq 0$ for $a \geq 0$ and $\omega(u) = 1$.

- Theorem: Let (V, C, u) be an order unit space. If $U \subseteq V$ is a subspace with $u \in U$, then $(U, C \cap U, u)$ is again an order unit space. Any probability measure σ on U can be extended to a probability measure ω on V, i.e. there is a probability measure $\omega : V \to \mathbb{R}$ such that $\omega_{|U} = \sigma$.

- Corollary: There is at least one probability measure on every order unit space.
  - Because there is always a probability measure on the one-dimensional subspace $U = \mathbb{F}u$, i.e. $\sigma(\lambda u) = \lambda$.

Proof strategy

- Theorem: A plausibility measure Pl on a test space (X, Σ) almost agrees with some probability measure μ iff it is Archimedean.
- Proof strategy: Construct an order unit space (V, C, u) containing a vector v_A representing each $A \in E(X, \Sigma)$ such that the cone ordering agrees with Pl, i.e.

$$\text{Pl}(A) \preceq \text{Pl}(B) \quad \Rightarrow \quad v_A \leq v_B.$$

- Use the existence of a probability measure on (V, C, u) to infer the existence of an almost agreeing probability measure on (X, Σ).

Order unit space from a plausibility measure

- Let V be the vector space over $\mathbb{Q}$ with orthonormal basis $\{e_x\}_{x \in X}$. The vector corresponding to $A \in E(X, \Sigma)$ is then $e_A = \sum_{x \in A} e_x$. In particular $e_\emptyset = 0$.
- Define the convex cone C to be the set of all finite, non-negative linear combinations of vectors of the form

$$e_A - e_B$$

  for all $A, B \in E(X, \Sigma)$ such that $\mathrm{Pl}(A) \succeq \mathrm{Pl}(B)$.
- Consider a test $T \in \Sigma$ and let $u = e_T$. We need to show that u is an order unit. This means checking
  1. $-u \not\geq 0$,
  2. For any $a \in V$, there is a $\lambda \in \mathbb{F}$ such that $\lambda u + a \geq 0$.
- 2 is fairly straightforward, so we focus on 1. This is where the Archimedean condition comes in.

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- Need to show that $-e_T$ is not a positive linear combination of vectors ofr the form $e_A - e_B$ for $\mathrm{Pl}(A) \succeq \mathrm{Pl}(B)$.
- This is a special case of: If $\mathrm{Pl}(A) \prec \mathrm{Pl}(B)$ then $e_A - e_B \notin C$.
  - Take $A = \emptyset$ and $B = T$.
- Assume $e_A - e_B \in C$. Then, there are events $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ such that

$$e_A - e_B = \sum_j \lambda_j (e_{A_j} - e_{B_j}),$$

where $\lambda_j \in \mathbb{Q}$ and $\mathrm{Pl}(A_j) \succeq \mathrm{Pl}(B_j)$.

$$e_A - e_B = \sum_j \lambda_j (e_{A_j} - e_{B_j})$$

- Because everything is rational, there is a positive integer k such that,

$$ke_A - ke_B = \sum_j r_j (e_{A_j} - e_{B_j}),$$

  where the r_j's are positive integers.

- Define $(A'_1, \ldots, A'_m)$ where the first r_1 elements are A_1,q the next r_2 are A_2, etc. and similarly for $(B'_1, \ldots, B'_m)$. Then

$$ke_A - ke_B = \sum_j (e_{A'_j} - e_{b'_j}),$$

$$\Rightarrow \sum_j e_{B'_j} + ke_A = \sum_j e_{A'_j} + ke_B.$$

Checking that u is an order unit

$$\sum_j e_{B'_j} + ke_A = \sum_j e_{A'_j} + ke_B$$

- Now construct the lists

$$(A'_1, \ldots, A'_m, B, \ldots, B) \qquad (B'_1, \ldots, B'_m, A, \ldots, A),$$

by appending k copies of B or A respectively. Then, each $x \in X$ occurs the same number of times in these lists.

- By construction, Pl$(A'_j) \succeq$ Pl(B'_j) and Pl$(B) \succ$ Pl(A).
- The Archimedean condition then gives Pl$(B) \preceq$ Pl(A), which is a contradiction.

Conclusion

Summary and Future Work

- Summary:
  - Plausibility measures can be defined for test spaces.
  - The conditions for almost agreement are the same as in the classical case.
  - The conditions for agreement are the same as the classical case for finite test spaces and more complicated for locally finite test spaces (including quantum).
- Future directions:
  - Is there an efficient algorithm for determining agreement?
  - Develop plausibilistic generalizations of quantum theory, e.g. is there a natural quantum theory on vector spaces over finite fields that makes more detailed predictions than Schumacher-Westmoreland theory?
  - Operational axioms for plausibilistic quantum theory.
  - Algorithms for plausibilistic inference in general theories.