What are quantum states?

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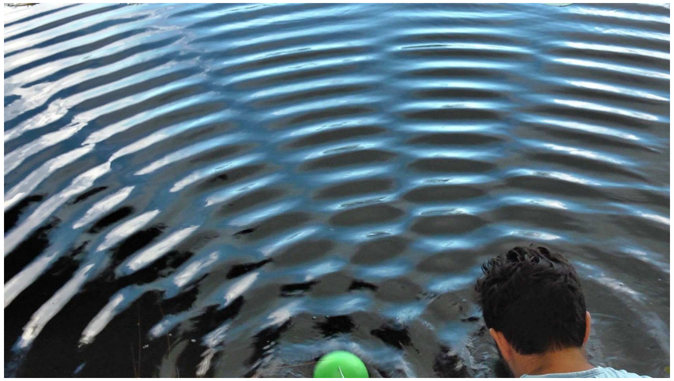
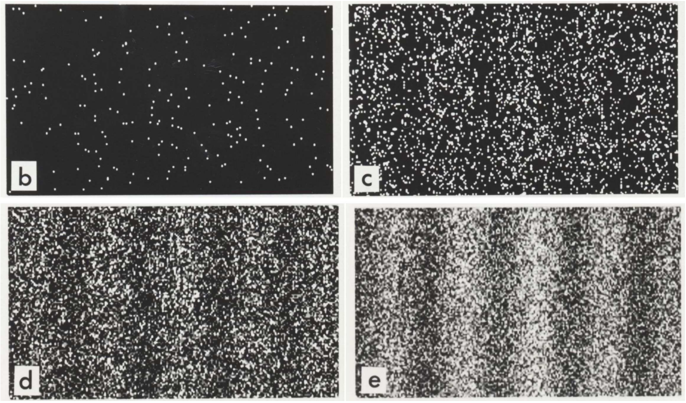
What are quantum states?

- *ψ-ontic view*: Quantum states are real, objective properties of quantum systems, akin to classical fields.



- *ψ-epistemic view*: Quantum states represent our knowledge or about quantum systems, akin to a classical probability distribution.



See ML, Quanta 3 pp. 67–155 (2014) for a review.

Overview

Review of quantum theory

Quantum Probability

Reality of the Quantum State

Conclusion

Review of quantum theory

Textbook quantum theory (finite dimensional version)

- A physical system A is associated with a Hilbert space $\mathcal{H}_A = \mathbb{C}^d$. (Pure) states of the system are unit vectors $|\psi\rangle \in \mathcal{H}_A$.
- A (nondegenerate) measurement is associated with an orthonormal basis

$$M = \{|a_1\rangle, |a_2\rangle, \cdots, |a_d\rangle\}.$$

The outcome a_j occurs with probability

$$\text{Prob}(a_j|\psi, M) = |\langle a_j|\psi\rangle|^2 .$$

- A system AB composed of two subsystems A and B is associated with the Hilbert space

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \text{span}\left(|\psi\rangle_A \otimes |\phi\rangle_B\right).$$

- More generally, the state of a system A is a positive operator ρ acting on $\mathcal{H}_A$ that satisfies Tr $(\rho) = 1$. The probability of obtaining outcome a_j in a measurement $\{|a\rangle_j\}$ is $\langle a_j| \rho |a_j\rangle$.
- Examples:
  - *Pure states*: Let $\rho = |\psi\rangle\langle\psi|$. Then,

$$|\langle a_j|\psi\rangle|^2 = \langle a_j|\psi\rangle \langle\psi|a_j\rangle = \langle a_j| \rho |a_j\rangle .$$

  - *Mixed states*: If $|\psi_k\rangle$ is prepared with probability p_k then let $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ and then

$$\sum_k p_k |\langle a_j|\psi_k\rangle|^2 = \sum_k p_k \langle a_j|\psi_k\rangle \langle\psi_k|a_j\rangle = \langle a_j| \rho |a_j\rangle .$$

- For a joint state ρ_{AB} on $\mathcal{H}_{AB}$, define the reduced state on A as

$$\rho_A = \text{Tr}_B\left(\rho_{AB}\right)$$

where, for an operator,

$$\rho_{AB} = \sum_{jklm} \alpha_{jk;lm} \left|j\right\rangle\left\langle k\right|_A \otimes \left|l\right\rangle\left\langle m\right|_B$$

$$\text{Tr}_B\left(\rho_{AB}\right) = \sum_{jkl} \alpha_{jk;ll} \left|j\right\rangle\left\langle k\right|_A .$$

- Then,

$$\sum_k \left\langle a_j\right| \otimes \left\langle b_k\right| \rho_{AB} \left|a_j\right\rangle \otimes \left|b_k\right\rangle = \left\langle a_j\right| \rho_A \left|a_j\right\rangle .$$

Quantum Probability

Classical	**Quantum**
Sample space $\Omega_A = \{a_1, a_2, \ldots\}$	Hilbert space $\mathcal{H}_A = \mathbb{C}^d$
Probability distribution $P(A = a_j) \geq 0$, $\sum_j P(A = a_j) = 1$	Density operator $\rho_A \in \mathfrak{L}^+(\mathcal{H}_A)$, $\text{Tr}_A(\rho_A) = 1$
Cartesian product $\Omega_A \times \Omega_B$	Tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$
Joint distribution $P(A, B)$	Bipartite state ρ_{AB}
Marginal distribution $P(B) = \sum_j P(A = a_j, B)$	Reduced state $\rho_B = \text{Tr}_A(\rho_{AB})$

For more details see ML and R. Spekkens, Phys. Rev. A 88 052130 (2013).

- Classically, the conditional probability distribution is defined as

$$P(B = b_k | A = a_j) = \frac{P(A = a_j, B = b_k)}{P(A = a_j)}.$$

- What should the quantum analog of this be?
  - $\rho_{B|A} = \rho_{AB}\rho_A^{-1}$?
  - $\rho_{B|A} = \rho_A^{-1}\rho_{AB}$?
- Neither of these is positive.

Quantum conditional states

- Define a family of positive products of positive operators

$$G \star^{(n)} H = \left(H^{\frac{1}{2n}} G^{\frac{1}{n}} H^{\frac{1}{2n}} \right)^n .$$

- Two important special cases:
  - $G \odot H = \lim_{n \to \infty} \left(G \star^{(n)} H \right) = e^{(\ln G + \ln H)}$
  - $G \star H = G \star^{(1)} H = H^{\frac{1}{2}} G H^{\frac{1}{2}}$
- Define conditional states:

$$\rho^{(n)}_{B|A} = \rho_{AB} \star^{(n)} \rho_A^{-1} .$$

  - Cerf-Adami: $\rho^{(\infty)}_{B|A} = \rho_{AB} \odot \rho_A^{-1}$
  - The $n = 1$ case: $\rho_{B|A} = \rho_{AB} \star \rho_A^{-1}$

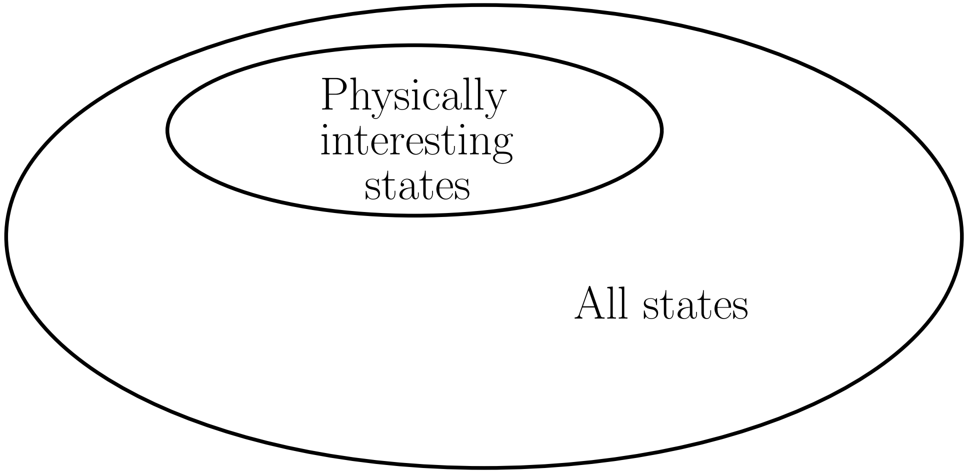
ML, Phys. Rev. A 74 042310 (2006). AIP Conference Proceedings 889 pp. 172–186 (2007).
ML & D. Poulin, Ann. Phys. 323 1899 (2008).
N. Cerf & C. Adami, Phys. Rev. Lett. 79 5194 (1997).

Representation

- Generic probability distribution over N variables: $O(d^N)$ params.
- Generic quantum state on N systems: $O(d^{2N})$ params.

Computation of marginals

- $P(A_1) = \sum_{A_2,A_3,\ldots,A_N} P(A_1, A_2, \ldots, A_N)$
- $\rho_{A_1} = \text{Tr}_{A_2 A_3 \ldots A_N} \left(\rho_{A_1 A_2 \ldots A_N}\right)$



Definition. A and B are conditionally independent given C if any of the following equivalent conditions holds:

- $P(A|B,C) = P(A|C)$
- $P(B|A,C) = P(B|C)$
- $P(A,B|C) = P(A|C)P(B|C)$
- $H(A:B|C) = 0,$

where

$$\begin{aligned} H(A:B|C) &= H(A|C) - H(A|B,C) \\ &= H(A,C) + H(B,C) - H(C) - H(A,B,C). \end{aligned}$$

and

$$H(X) = -\sum_X P(X) \log P(X).$$

Definition. A and B are conditionally independent given C if $S(A:B|C) = 0$, where

$$S(A:B|C) = S(A,C) + S(B,C) - S(C) - S(A,B,C)$$

$$S(X) = -\mathrm{Tr}_X\left(\rho_X \log \rho_X\right).$$

Theorem. *If* $S(A:B|C) = 0$ *then*

- $\rho^{(n)}_{A|BC} = \rho^{(n)}_{A|C}$
- $\rho^{(n)}_{B|AC} = \rho^{(n)}_{B|C}$
- $\rho^{(n)}_{AB|C} = \rho^{(n)}_{A|C}\rho^{(n)}_{B|C}$.

- For $\odot$ all converse implications hold.
- For $\star$ first two converse implications hold.

ML & D. Poulin, Ann. Phys. 323 1899 (2008).

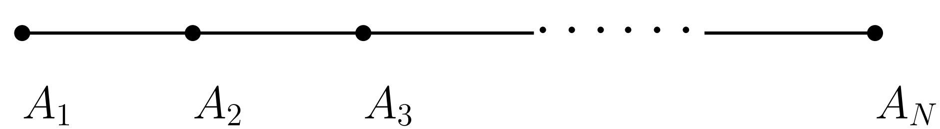
Quantum Markov Chains

- A general state of N systems can be written as

$$\rho_{A_1,A_2,\ldots,A_N} = \rho^{(n)}_{A_N|A_1A_2\ldots A_{N-1}} \star^{(n)} \ldots \star^{(n)} \rho^{(n)}_{A_3|A_2A_1} \star^{(n)} \rho^{(n)}_{A_2|A_1} \star^{(n)} \rho_{A_1}.$$

- Imposing the constraint $S(A_j : A_1A_2\ldots A_{j-2}|A_{j-1}) = 0$ gives

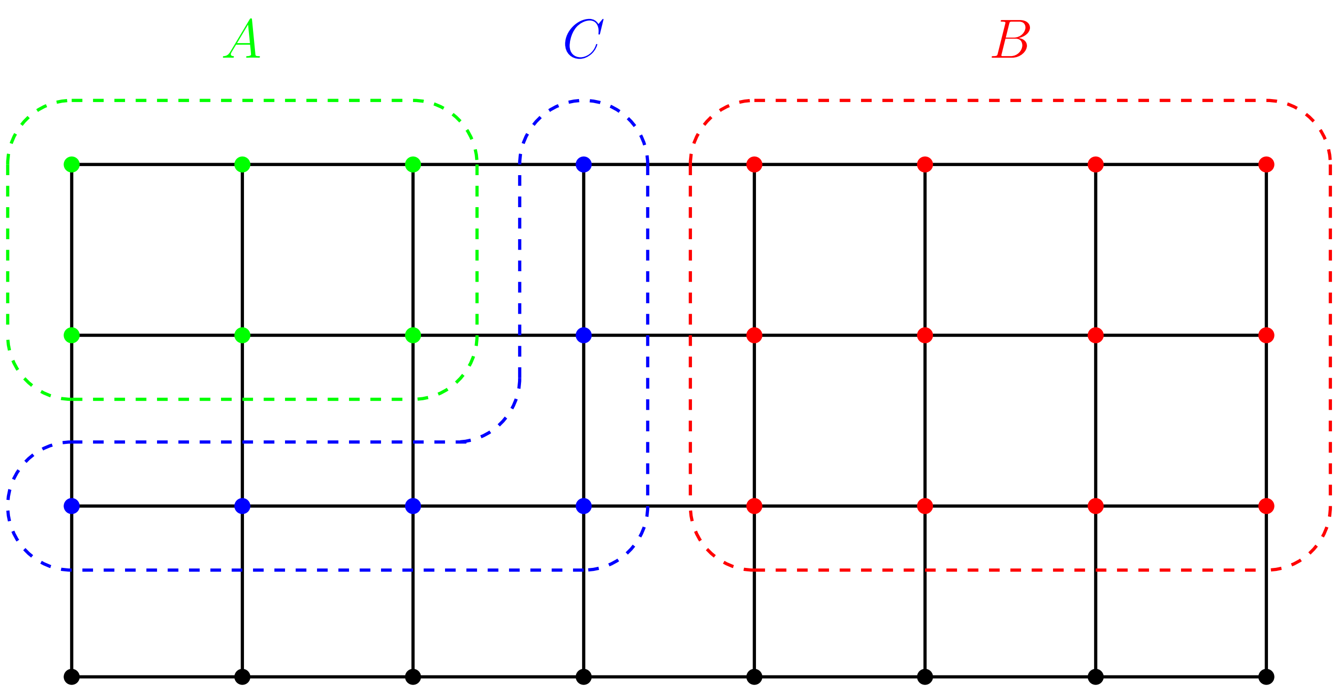
$$\rho_{A_1,A_2,\ldots,A_N} = \rho^{(n)}_{A_N|A_{N-1}} \star^{(n)} \ldots \rho^{(n)}_{A_3|A_2} \star^{(n)} \rho_{A_2|A_1} \star^{(n)} \rho_{A_1}$$



- This decomposition and the one that follows can be used in a quantum generalization of *belief propagation* algorithms.

ML & D. Poulin, Ann. Phys. 323 1899 (2008).

Definition. A *Quantum Markov Network* (G, ρ) is an undirected graph $G = (V, E)$, where the vertices are quantum systems, and a density operator ρ_V that satisfies $S(A : B|C) = 0$ for all disjoint $A, B, C \subseteq V$ such that every path from A to B intersects C.



ML & D. Poulin, Ann. Phys. 323 1899 (2008).

Quantum Hammersley-Clifford Theorem

Theorem. *If (G, ρ) is a Quantum Markov Network and ρ is strictly positive then*

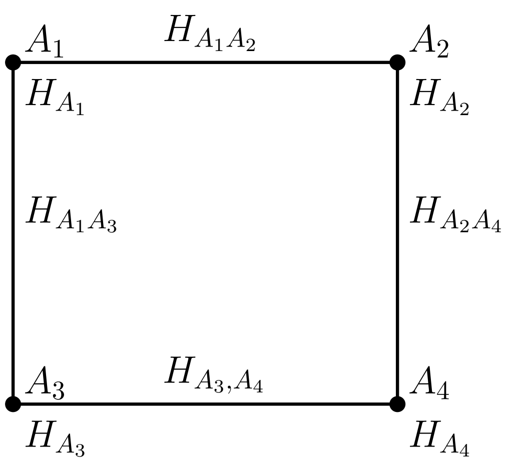
$$\rho_V = \frac{1}{Z} \odot_{C \in \mathfrak{C}} \nu_C,$$

where $\mathfrak{C}$ is the set of cliques in G.

- Alternatively,

$$\rho_V = \frac{1}{Z} e^{-\beta \sum_{C \in \mathfrak{C}} H_C},$$

where $H_C = -\frac{1}{\beta} \ln \nu_C$.



- Converse does not hold: there are extra constraints on the local Hamiltonians.

ML & D. Poulin, Ann. Phys. 323 1899 (2008).

Applications to numerical simulation of quantum systems

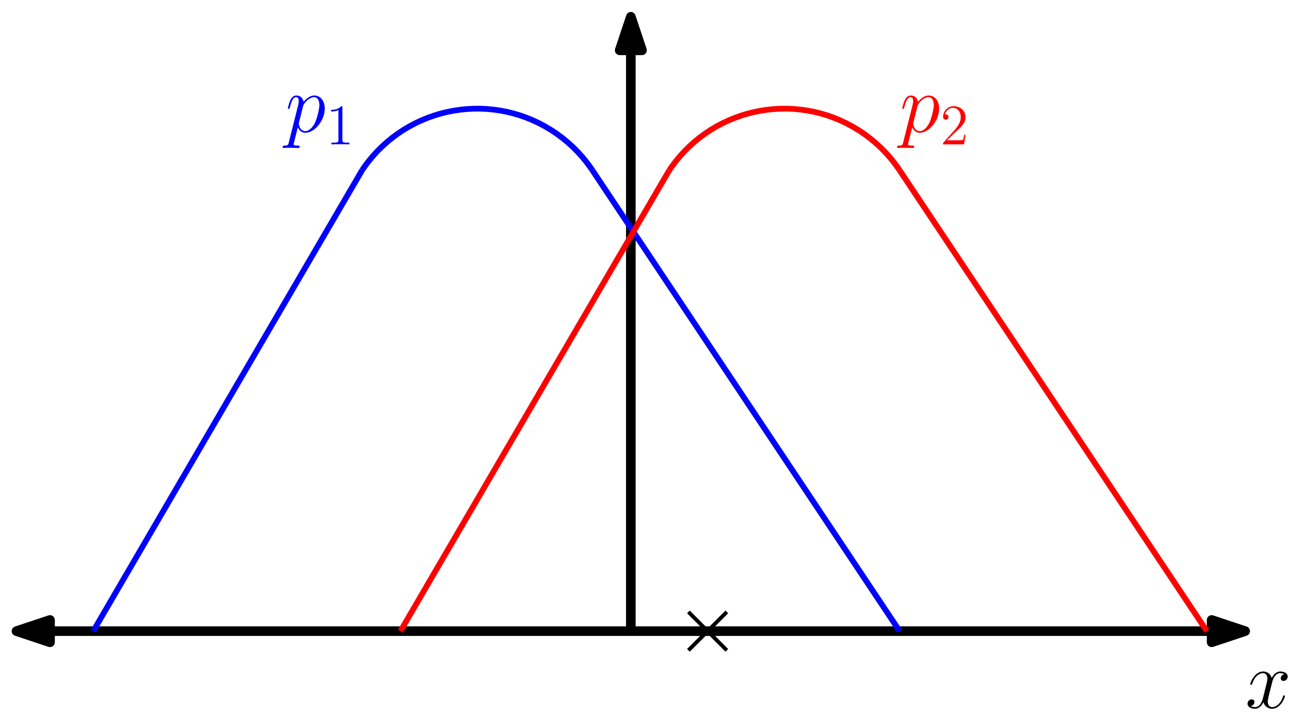
- Applications of Quantum Belief Propagation:
  - Statistical mechanics of quantum Ising spin chains and spin glasses:
    - E. Bilgin and D. Poulin, Phys. Rev. B 81 054106 (2010).
    - C. Laumann, A. Scardicchio and S. L. Sondhi, Phys. Rev. B 78 134424 (2008).
    - D. Nagaj, E. Farhi, J. Goldstone, P. Shor and I. Sylvester, Phys. Rev. B 77 214431 (2008).
  - Study of the connection between the quantum generalization of satisfiability and phase transitions:
    - C. Laumann, R. Moessner, A. Scardicchio and S. L. Sondhi, Quant. Inf. and Comp. vol. 10(1) pp. 1–15 (2010).
- Markov entropy decomposition (dual to belief propagation):
  - Used to obtain lower bounds on the free energy.
    - D. Poulin and M. Hastings, Phys. Rev. Lett. 106 080403 (2011).
    - A. J. Ferris and D. Poulin, Phys. Rev. B 87 205126 (2013).

Further work and applications

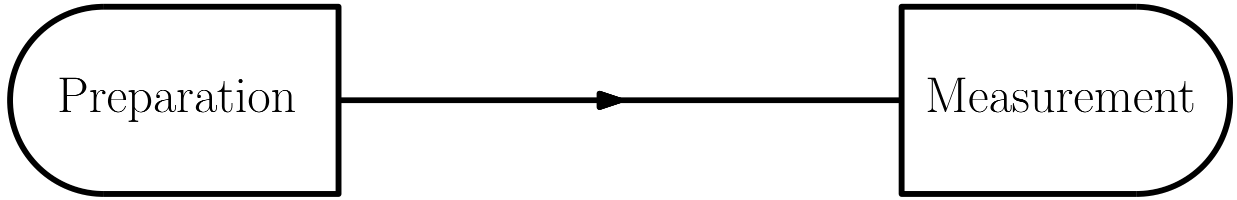
- ML and R. Spekkens, Phys. Rev. A 88 052130 (2013).
  - Unified formalism for preparations, measurements and dynamics
  - Quantum Bayes theorem
  - Retrodictive quantum theory
  - Quantum steering
- ML and R. Spekkens, J. Phys. A 47 275301 (2014).
  - Quantum sufficient statistics
  - Quantum state compatibility
  - Quantum state improvement and pooling
- B. Coecke & R. Spekkens, Synthese 186 651 (2012).
  - Category theoretic version of quantum Bayesian inference.
- E. G. Cavalcanti & R. Lal (2013). arXiv:1311.6852.
  - Used to analyse quantum generalization of Bell's locality condition.
- J. Norton (2014). `http://bit.ly/1km1Q4L`.
  - Quantum inductive logic

Reality of the Quantum State

Probability distributions can overlap



Prepare-and-measure experiments: Quantum description

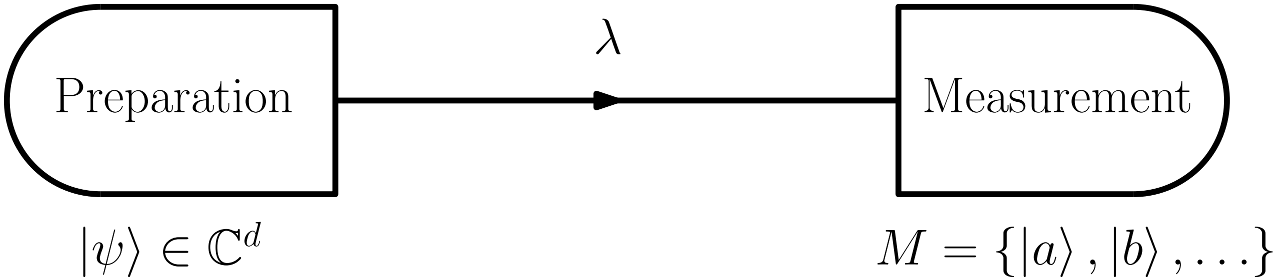


$|\psi\rangle \in \mathbb{C}^d$

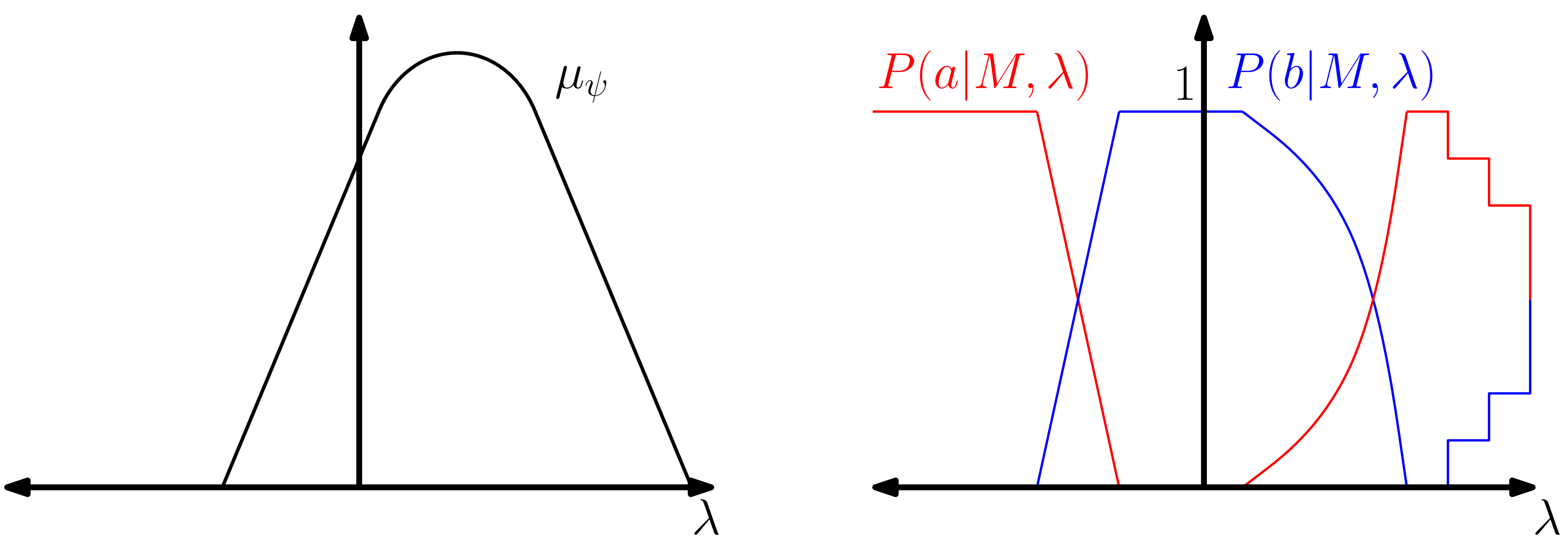
$M = \{|a\rangle\,, |b\rangle\,, \ldots\}$

$$\mathrm{Prob}(a|\psi, M) = |\langle a|\psi\rangle|^2$$

Prepare-and-measure experiments: Ontological description



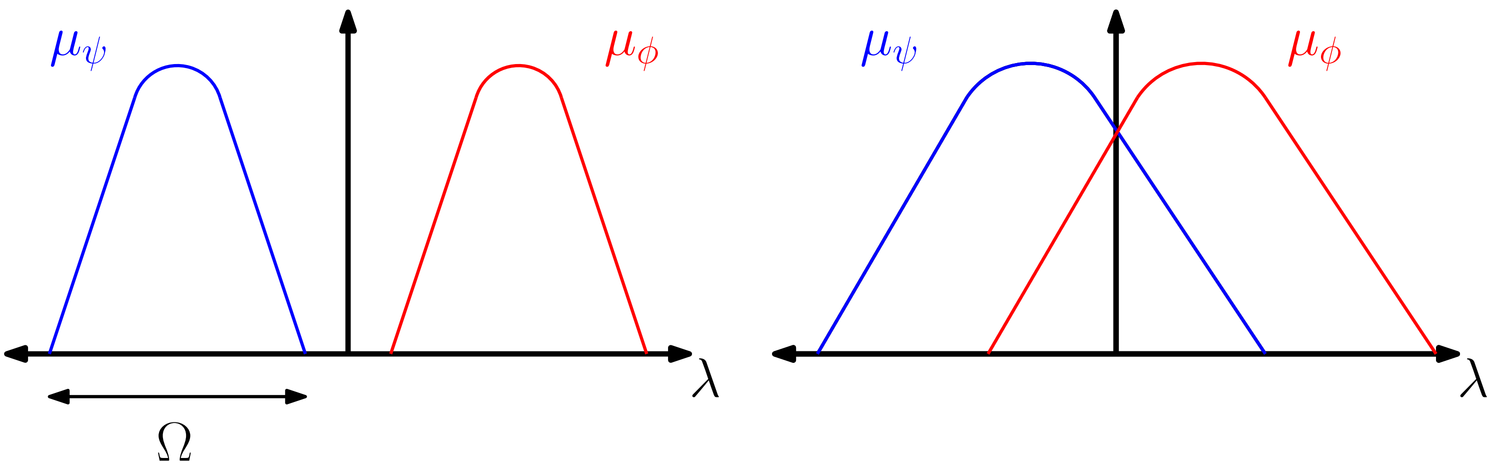
$$\text{Prob}(a|\psi, M) = |\langle a|\psi\rangle|^2$$



$$\text{Prob}(a|\psi, M) = \int P(a|M,\lambda) d\mu_\psi$$

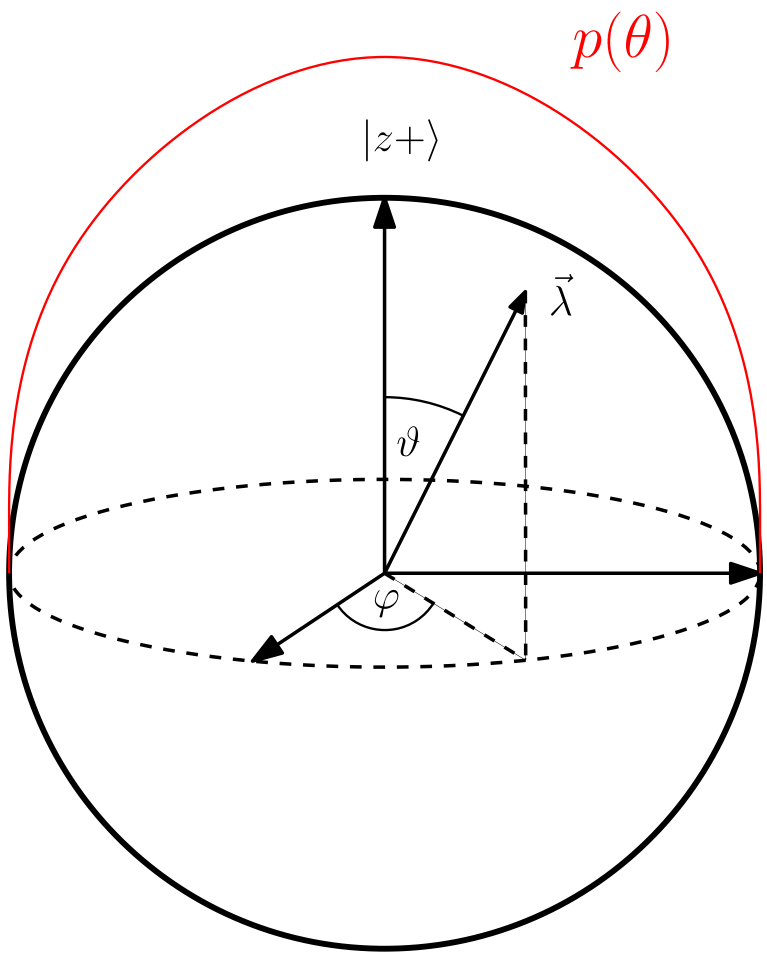
- $|\psi\rangle$ and $|\phi\rangle$ are *ontologically distinct* in an ontological model if there exists $\Omega \in \Sigma$ s.t.

$$\mu_\psi(\Omega) = 1 \qquad \mu_\phi(\Omega) = 0.$$



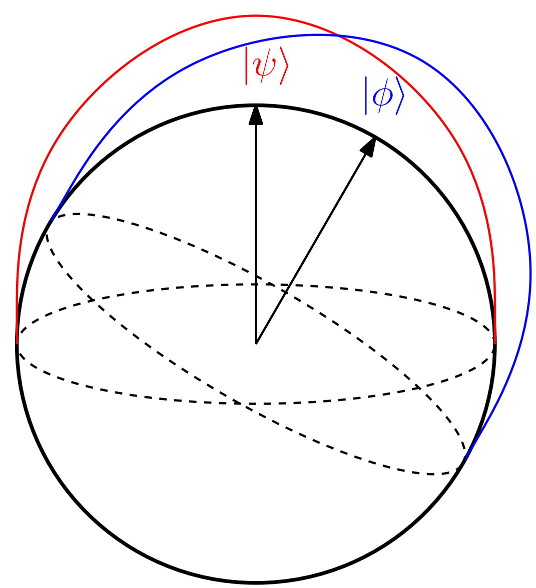
- An ontological model is *ψ-ontic* if every pair of states is ontologically distinct. Otherwise it is *ψ-epistemic*.

- The Colbeck-Renner theorem: R. Colbeck and R. Renner, arXiv:1312.7353 (2013).
- Hardy's theorem: L. Hardy, *Int. J. Mod. Phys. B*, 27:1345012 (2013) arXiv:1205.1439
- The Pusey-Barrett-Rudolph theorem: M. Pusey et. al., *Nature Physics*, 8:475–478 (2012) arXiv:1111.3328



$$\mu_{z+}(\Omega) = \int_\Omega p(\vartheta)\sin\vartheta d\vartheta d\varphi$$

$$p(\vartheta) = \begin{cases} \frac{1}{\pi}\cos\vartheta, & 0 \leq \vartheta \leq \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \vartheta \leq \pi \end{cases}$$



S. Kochen and E. Specker, *J. Math. Mech.*, 17:59–87 (1967)

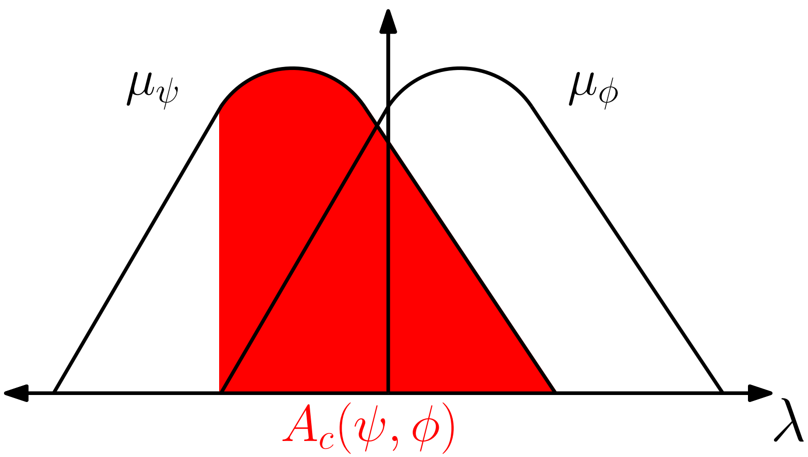
Models for arbitrary finite dimension

- Lewis et. al. provided a ψ-epistemic model for all finite d.
  - P. G. Lewis et. al., *Phys. Rev. Lett.* 109:150404 (2012) arXiv:1201.6554
- Aaronson et. al. provided a similar model in which every pair of nonorthogonal states is ontologically indistinct.
  - S. Aaronson et. al., *Phys. Rev. A* 88:032111 (2013) arXiv:1303.2834
- These models have the feature that, for a fixed inner product, the amount of overlap decreases with d.

Asymmetric overlap

- *Classical asymmetric overlap*:

$$A_c(\psi,\phi) := \inf_{\left\{\Omega\in\Sigma \middle| \mu_\phi(\Omega)=1\right\}} \mu_\psi(\Omega)$$



- An ontological model is *maximally ψ-epistemic* if

$$A_c(\psi,\phi) = |\langle\phi|\psi\rangle|^2$$

Main result

- Let $\mathcal{D} = \{|\phi_j\rangle\}_{j=1}^N$ be a set of quantum states and let $|\psi\rangle$ be any other quantum state. Define:

$$\bar{k}_{\mathcal{D}}(\psi) = \frac{\sum_{j=1}^N A_c(\psi, \phi_j)}{\sum_{j=1}^N |\langle\phi_j|\psi\rangle|^2}.$$

- We can construct a set of states in $\mathbb{C}^d$ such that

$$k_{\mathcal{D}}(\psi) \leq 2de^{-cd}.$$

ML, Phys. Rev. Lett. 112:160404 (2014)

- Review articles:
  - ML, Quanta 3 pp. 67–155 (2014).
  - D. Jennings and ML, arXiv:501.03202, to appear in Contemp. Phys. (2015).
- Contextuality and overlap bounds:
  - ML and O. Maroney, Phys. Rev. Lett. 110:120401 (2013).
  - ML, Phys. Rev. Lett. 112:160404 (2014).

Conclusion

Conclusions

- Quantum states are most fruitfully understood as states of knowledge, akin to classical probability distributions.
- However, we cannot straightforwardly understand quantum states as representing classical uncertainty about some true underlying state of reality.
- This suggests exploring more exotic ontologies that support a nonclassical probability theory, e.g.
  - Retrocausality
  - Relationalism
  - Many-worlds
  - Nonclassical logic

Future Directions

- Quantum probability:
  - Develop a quantum theory of Bayesian inference without a priori causal structure.
  - Develop quantum generalizations of probabilistic machine learning structures and algorithms.
  - Investigate monogamy of conditional states and applications, e.g. to simulation of many-body systems.
- Ontological models:
  - Find experimentally testable overlap bounds with low $k_{\mathcal{D}}(\psi)$.
  - Develop qinfo. applications, e.g. to communication complexity.
  - Investigate exotic ontologies that may close the explanatory gaps demonstrated by no-go theorems, e.g. retrocausality.

Additional slides

- Classical entropy is given by

$$H(A) = -\sum_A P(A) \ln P(A),$$

and conditional entropy by

$$H(B|A) = H(A,B) - H(A) = -\sum_{A,B} P(A,B) \ln P(B|A).$$

- Quantum entropy is given by

$$S(A) = -\text{Tr}\left(\rho_A \ln \rho_A\right),$$

and conditional entropy by

$$S(B|A) = S(A,B) - S(A) = -\text{Tr}\left(\rho_{AB} \ln \rho_{B|A}^{(\infty)}\right).$$

N. Cerf & C. Adami, Phys. Rev. Lett. 79 5194 (1997).

What is special about $\star$ and $\rho_{B|A}$?

- A conditional probability distribution $P(B|A)$ can be defined as a positive function on $\Omega_A \times \Omega_B$ that satisfies

$$\sum_B P(B|A) = 1.$$

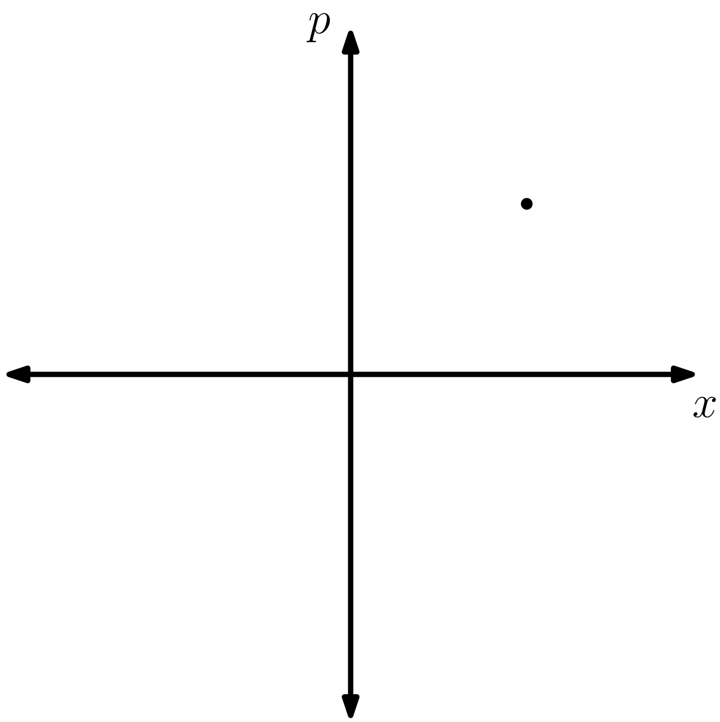
- A quantum conditional state $\rho_{B|A}$ with the $\star$-product can be defined as a positive operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ that satisfies

$$\text{Tr}_B \left(\rho_{B|A}\right) = I_A.$$

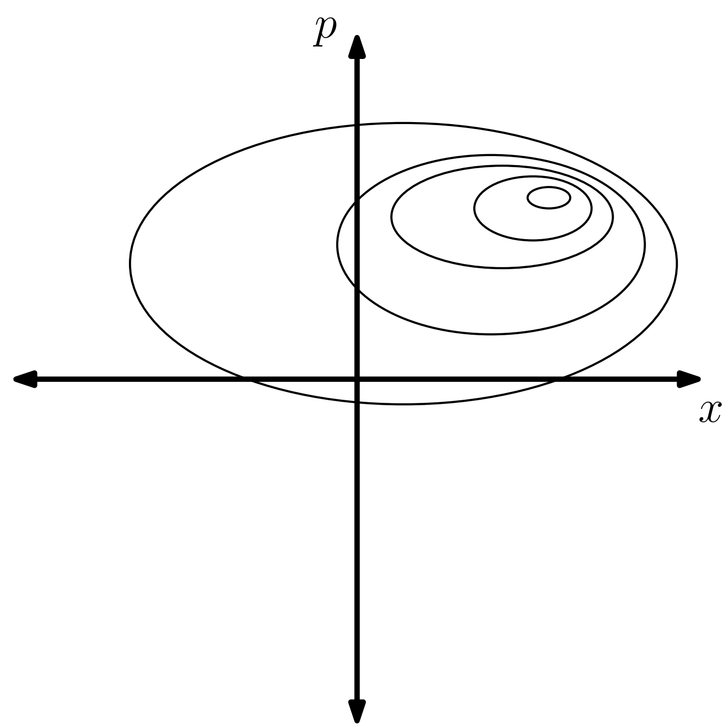
ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

Classical states

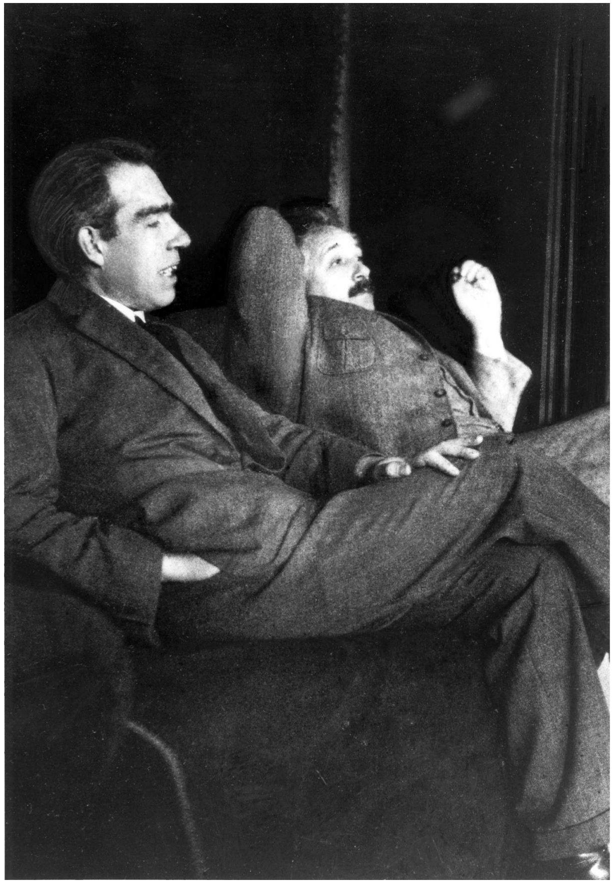
Ontic state



Epistemic state



Bohr and Einstein: ψ-epistemicists



Source: http://en.wikipedia.org/

> There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature. — Niels Bohr[a]

> [t]he ψ-function is to be understood as the description not of a single system but of an ensemble of systems. — Albert Einstein[b]

[a] Quoted in A. Petersen, "The philosophy of Niels Bohr", *Bulletin of the Atomic Scientists* Vol. 19, No. 7 (1963)

[b] P. A. Schilpp, ed., *Albert Einstein: Philosopher Scientist* (Open Court, 1949)

Penrose: ψ-ontologist

It is often asserted that the state-vector is merely a convenient description of 'our knowledge' concerning a physical system—or, perhaps, that the state-vector does not really describe a single system but merely provides probability information about an 'ensemble' of a large number of similarly prepared systems. Such sentiments strike me as unreasonably timid concerning what quantum mechanics has to tell us about the *actuality* of the physical world. — Sir Roger Penrose[1]

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[1] R. Penrose, *The Emperor's New Mind* pp. 268–269 (Oxford, 1989)

Interpretations of quantum theory

	ψ-epistemic	ψ-ontic
Copenhagenish	Copenhagen neo-Copenhagen (e.g. QBism, Peres, Zeilinger, Healey)	
Realist	Einstein Ballentine? Spekkens ?	Dirac-von Neumann Many worlds Bohmian mechanics Spontaneous collapse Modal interpretations

Experiments

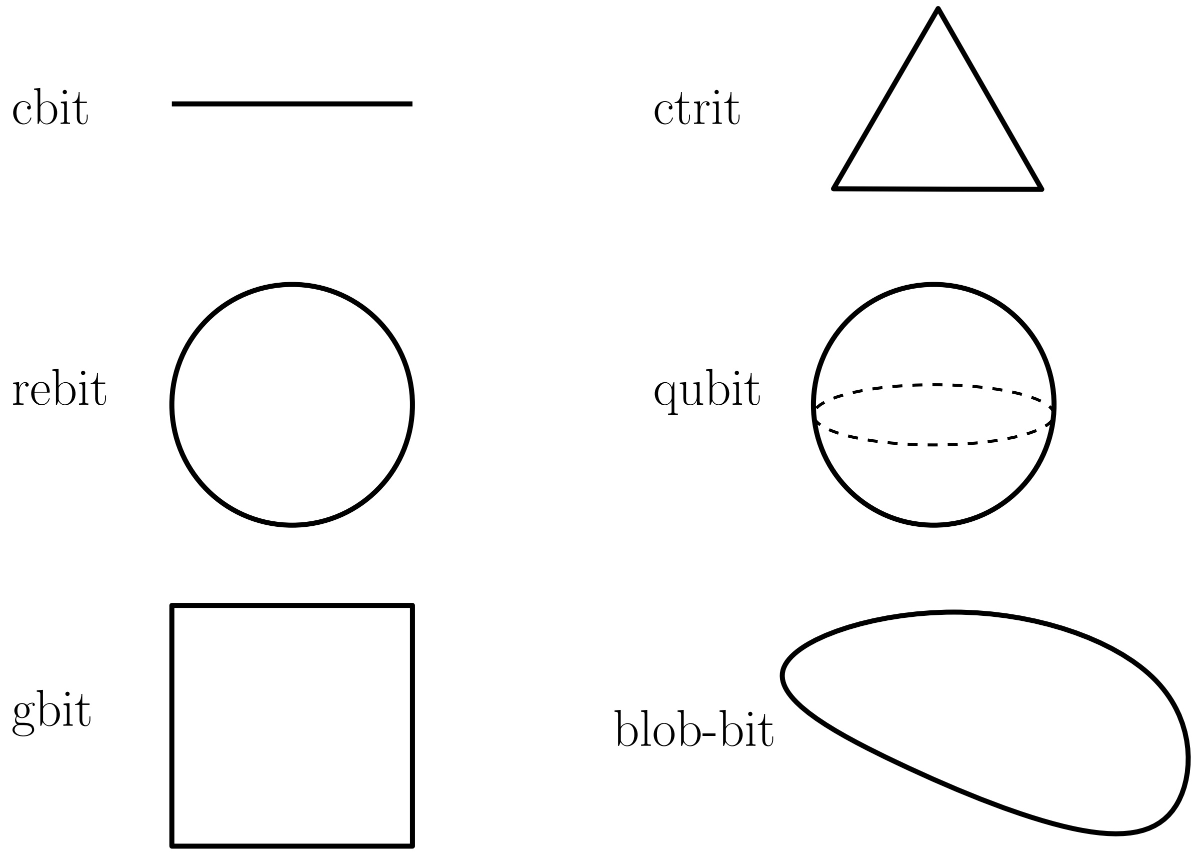
- Ringbauer et. al. obtained

$$k_{\mathcal{D}}(\psi) \leq 0.690 \pm 0.001$$

  in an optical system for $d = 4$.

- Ringbauer et. al. experiments required a fair sampling assumption and estimated $\approx 98\%$ detector efficiency required to do with out.
- Values close to zero are needed to convincingly rule out ψ-epistemic theories.
- Since we now know these results can be derived from noncontextuality inequalities, we can now search for optimal experiments.

- General framework for probabilistic theories that includes classical probability, quantum theory, PR-boxes, . . . as special cases.
- State space of a system is an arbitary compact convex set.

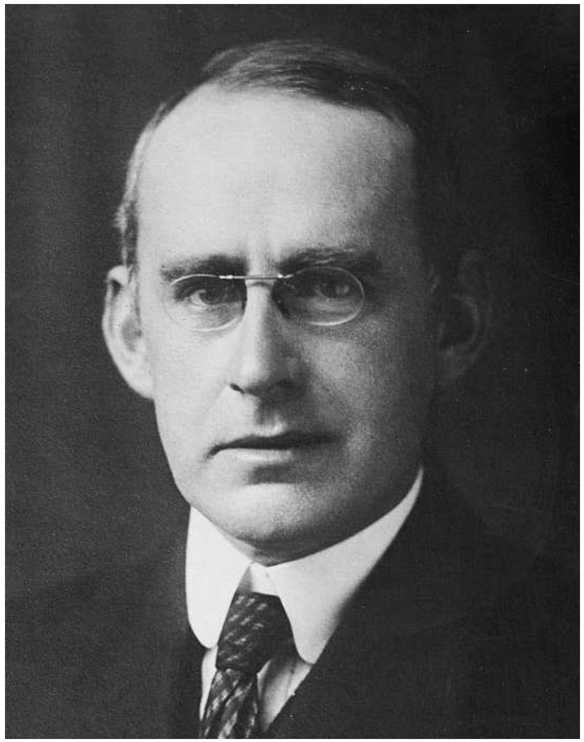


Applications of COTs

- Identifying the logical structure of information processing
  - Connection between cloning, broadcasting and distinguishability[2].
  - Nonclassicality + No entanglement $\Rightarrow$ Bit commitment[3].
  - de Finetti theorem[4].
  - Requirements for teleportation[5].
- Axiomatic reconstructions of quantum theory
  - L. Hardy, arXiv:quant-ph/0101012, arXiv:1104.2066.
  - B. Dakic, C. Brukner, in H. Halvorson (ed.) *Deep Beauty*, pp. 365–392 (CUP, 2011).
  - L. Masanes, M. Müller, New. J. Phys. 13:063001 (2011).
  - G. Chiribella, G. M. D'Ariano, P. Perinotti, Phys. Rev. A. 84:012311 (2011).

[2] H. Barnum, J. Barrett, ML, A. Wilce, Phys. Rev. Lett. 99:240501 (2007).
[3] H. Barnum, O. Dahlsten, ML, B. Toner, Proc. IEEE Info. Theory Workshop, 2008, pp. 386–390.
[4] J. Barrett, ML, New J. Phys. 11:033024 (2009).
[5] H. Barnum, J. Barrett, ML, A. Wilce, Proc. Clifford Lectures 2008 (2012).

Supremacy of the Second Law



George Grantham Bain Collection (Library of Congress)

The law that entropy always increases, holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations then so much the worse for Maxwell's equations. If it is found to be contradicted by observation well, these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation. — Sir Arthur Eddington[a]

[a] *The Nature of the Physical World* (Cambridge University Press, 1929) p. 74.

The Resource Theory of (Classical) Nonuniformity

- Thermodynamics can be formulated as a *resource theory*. If $H =$ const. then this reduces to the theory of *nonuniformity*[6] .

- States: Probability distributions $\boldsymbol{p}$.

- Free operations:
  - Reversible transformations
  - Adding uniform ancillas $(\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d})$.
  - Discarding subsystems.

- Second law: If $\boldsymbol{p} \to \boldsymbol{p}'$ is possible under free operations (with $\boldsymbol{p}$, $\boldsymbol{p}'$ defined on the same space) then

$$S(\boldsymbol{p}') \geq S(\boldsymbol{p}).$$

[6] G. Gour, M. Müller, V. Narasimachar, R. Spekkens, N. Halpern, arXiv:1309.6586.

- For an arbitrary COT, this cannot be formulated so easily.
- States: Elements ω of a convex set.
- Free operations:
  - Reversible transformations (automorphism group)
  - *Adding maximally mixed ancillas?*
    - Generally there is no unique notion of a uniform state.
  - Discarding subsystems.
- *Second law?*
  - Although some entropy functions have been proposed[7], it is not clear whether they are relevant to thermodynamics, or indeed if there is a unique thermodynamic entropy at all.

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[7] H. Barnum, J. Barrett, L. Clark, ML, R. Spekkens, N. Stepanik, A. Wilce, R. Wilke, New J. Phys. 12:033024 (2010). A. Short, S. Wehner, New J. Phys. 12:033023 (2010).

- We can consider hybrid theories in which we can have both classical and COT systems.
- States: Elements $\boldsymbol{p} \otimes \omega$ of the joint state space.
- Free operations:
  - Reversible transformations (automorphism group)
  - *At least, we should be able to add uniform classical ancillas* $\left(\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}\right)$.
  - Discarding subsystems.
- *Second Law: At least we expect that if $\boldsymbol{p} \otimes \omega \to \boldsymbol{p}' \otimes \omega$ is possible under free operations (with $\boldsymbol{p}$, $\boldsymbol{p}'$ defined on the same space) then*

$$S(\boldsymbol{p}') \geq S(\boldsymbol{p}).$$

Proposed Axioms for Quantum Theory

1. Automorphism group is transitive.
2. von Neumann's assumption.
3. Second Law for classical systems.

- What I know so far:
  - Rules out polygons with even number of sides in 2D.
  - There are non-classical and non-quantum theories that satisfy the axioms, e.g. hyperspheres.
- Conjecture: Axioms single out state spaces of Jordan algebras.