# Plausibility Measures on Test Spaces

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## Introduction

Recent years have seen a growth of interest in *possibilistic*, *modal*, or *relational* theories [1, 2], in which the continuum of of probability assignments is replaced by a two valued assessment of whether or not an outcome is possible.
If we are serious about using weaker predictive structures in physics, we need to employ a structure that is capable of reproducing probabilistic predictions where they are known to work well, but allows for weaker predictions in general.
Classically, *plausibility measures* [3] are such a structure. Both possibility and probability measures are examples of plausibility measures, and in general they allow for qualitative comparisons, e.g. $A$ is more likely than $B$, without requiring precise numerical probabilities.
Here, we generalize plausibility measures to test spaces, which can be used to represent nonclassical theories such as quantum theory. We address the question of when a plausibility measure *agrees* with a probability measure, i.e. when its comparative relations are rich enough to be faithfully represented by a probability measure.

## Test Spaces

**Definition: Test space**

A *test space* $(X, \Sigma)$ consists of a set $X$ of *outcomes* together with a set of subsets $\Sigma \subseteq 2^X$ such that $\Sigma$ covers $X$, i.e. $\bigcup_{T\in\Sigma} T = X$.

A test $T \in \Sigma$ represents a measurement that can be performed on the system. Distinct tests may overlap, i.e. they can share outcomes in common.

**Definition: Locally finite**

A test space is *locally finite* if every test $T \in \Sigma$ is a finite set.

All test spaces are assumed locally finite in what follows.

**Definition: Events**

A subset $A \subseteq X$ is an *event* if it is a subset of a test. $E(X, \Sigma)$ denotes the set of events.

### Examples

- Finite classical test space: If $X$ is finite and $\Sigma = \{X\}$, then $E(X, \Sigma) = 2^X$ and $(X, E(X, \Sigma))$ is a finite classical sample space.
- The Specker triangle: $X = \{x, y, z\}$ and $\Sigma = \{\{x, y\}, \{y, z\}, \{z, x\}\}$. This has $E(X, \Sigma) = \{\emptyset, \{x\}, \{y\}, \{z\}.\{x, y\}, \{y, z\}, \{z, x\}\}$.
- Finite-dimensional quantum test space: $X =$ the set of rays in a Hilbert space $\mathcal{H}$, $\Sigma =$ the set of orthonormal bases.

## Probability and Plausibility Measures

**Definition: Probability measure**

A *probability measure* on a test space $(X, \Sigma)$ is a function $\mu : X \to \mathbb{R}_{\geq 0}$ such that $\sum_{x\in T} \mu(x) = 1$ for every test $T \in \Sigma$.

We use the same notation for the additive extension $\mu : E(X, \Sigma) \to \mathbb{R}_{\geq 0}$, i.e. $\mu(A) = \sum_{x\in A} \mu(x)$.

**Definition: state space**

The set of probability measures on a test space $(X, \Sigma)$ is an affine subspace of $\mathbb{R}^X$ called the *state space* of $(X, \Sigma)$.

### Examples

- Finite classical test space: This gives the usual definition of a probability measure. State space is the probability simplex with dimension $|X| - 1$.
- Finite-dimensional quantum test space: By Gleason's theorem, for $\dim(\mathcal{H}) \geq 3$, probability measures are of the form $\mu(x) = \langle x|\rho|x\rangle$ for some density operator $\rho$ on $\mathcal{H}$. State space has dimension $\dim(\mathcal{H})^2 - 1$.

**Definition: Plausibility measure**

A *plausibility measure* on a test space $(X, \Sigma)$ is a function $\mathrm{Pl} : E(X, \Sigma) \to D$, where $(D, \preceq, 0, 1)$ is a bounded poset with minimal element 0 and maximal element 1, satisfying

- $\mathrm{Pl}(\emptyset) = 0$.
- For all $T \in \Sigma$, $\mathrm{Pl}(T) = 1$.
- If $A$ and $B$ are events with $A \subseteq B$, then $\mathrm{Pl}(A) \preceq \mathrm{Pl}(B)$.

### Examples

- A *possibility measure* is a plausibility measure with $D = \{0, 1\}$, with 0 interpreted as impossibility and 1 as possibility.
- A probability measure is a special case of a plausibility measure with $D = [0, 1]$ and the usual ordering.
- A set $\{\mu_j\}$ of probability measures can be used to define a plausibility measure with $\mathrm{Pl}(A) \preceq \mathrm{Pl}(B)$ iff $\mu_j(A) \leq \mu_j(B)$ for all $j$.

## Agreement

**Definition: Agreement**

A plausibility measure Pl on a test space $(X, \Sigma)$ *agrees* with a probability measure $\mu$ if

$$\mathrm{Pl}(A) \preceq \mathrm{Pl}(B) \iff \mu(A) \leq \mu(B).$$

Pl *almost agrees* with $\mu$ if

$$\mathrm{Pl}(A) \preceq \mathrm{Pl}(B) \implies \mu(A) \leq \mu(B)$$

Agreement implies that the image of Pl is totally ordered. Almost agreement is much weaker as it allows partial ordering and also allows $\mu(A) = 0$ when $\mathrm{Pl}(A) \succ 0$.

### Example

- Finding a probability measure that agrees with a plausibility measure is not always possible. For example, the Specker triangle has a possibility measure with $\mathrm{Pl}(\{x\}) = 0$, $\mathrm{Pl}(\{y\}) = 1$ and $\mathrm{Pl}(\{z\}) = 1$, but there exists no probability measure with $\mu(x) = 0$, $\mu(y) > 0$ and $\mu(z) > 0$. This is because we must have $\mu(x)+\mu(y) = 1$ and $\mu(y)+\mu(z) = 1$, but these assignments imply $\mu(x) + \mu(y) = 0 + 1 - \mu(z) < 1$.

## The Archimedean Condition

To obtain agreement, we need a minimal order theoretic counterpart of the additivity axiom for probability measures. This is the Archimedean condition.

**Definition: Archimedean condition**

A plausibility measure Pl on a test space $(X, \Sigma)$ is *Archimedean* if, for every pair $(A_1, A_2, \ldots, A_n)$ and $(B_1, B_2, \ldots, B_n)$ of families of events such that every outcome in $X$ occurs the same number of times in both,

$$\mathrm{Pl}(A_1) \preceq \mathrm{Pl}(B_1), \quad \ldots, \quad \mathrm{Pl}(A_{n-1}) \preceq \mathrm{Pl}(B_{n-1}) \quad \Rightarrow \quad \mathrm{Pl}(A_n) \succeq \mathrm{Pl}(B_n).$$

### Example

- The plausibility measure on the Specker triangle with $\mathrm{Pl}(\{x\}) = 0$, $\mathrm{Pl}(\{y\}) = 1$, $\mathrm{Pl}(\{z\}) = 1$ fails the Archimedean condition. Consider the families $(\{y, z\}, \{x, z\}, \{x\})$ and $(\{x, y\}, \{x, z\}, \{z\})$ in which $x$, $y$ and $z$ appear the same number of times. We have $\mathrm{Pl}(\{y, z\}) \preceq \mathrm{Pl}(\{x, y\})$ and $\mathrm{Pl}(\{x, z\}) \preceq \mathrm{Pl}(\{x, z\})$, but $\mathrm{Pl}(\{x\}) \prec \mathrm{Pl}(\{z\})$.

## Main Results

**Theorem: Almost agreement**

A plausibility measure Pl on a locally finite test space almost agrees with some probability measure $\mu$ if Pl is Archimedean.

**Theorem: Agreement**

A plausibility measure Pl on a locally finite test space with a finite dimensional state space agrees with some probability measure $\mu$ iff the image of Pl is totally ordered and Pl is Archimedean.

The proofs of these theorems are based on Hahn-Banach theorems for order unit spaces over the rational field. See the paper [4] for details.
The second of these theorems generalizes the classical result that a plausibility measure on a finite sample space agrees with a probability measure iff its image is totally ordered and it is Archimedean [5].

## Open Questions

- What are the minimal conditions on a locally finite test space for agreement, e.g. can finite dimensionality be replaced by topological conditions?
- Extension to non-locally finite test spaces.
- Can we unify possibilistic approaches to no-go theorems [2] with the usual probabilistic approach using plausibility measures?
- Can we develop quantum Bayesian networks and inference algorithms using plausibility measures (this was one of the original motivations in the classical case [6])?
- Are weaker predictive structures such as plausibility measures relevant for quantum gravity?

### References

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