Aharonov meets Spekkens: What do quantum logical pre- and post-selection paradoxes tell us about the nature of reality?

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Collaborator

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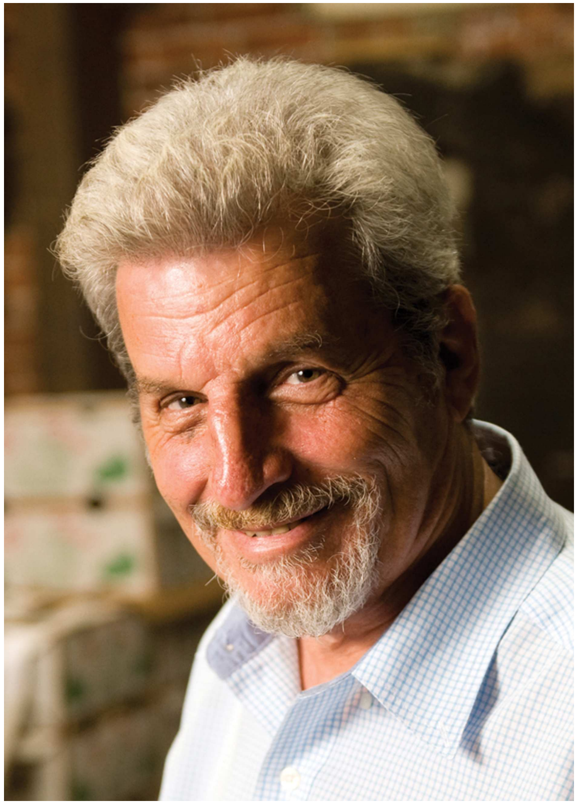
Aharonov



- "Progress through paradox"[a]:
  - ☐ Three box paradox
  - ☐ Quantum pigeonhole principle
  - ☐ Quantum Cheshire cats
  - ☐ Anomalous weak values
  - ☐ Protective measurement

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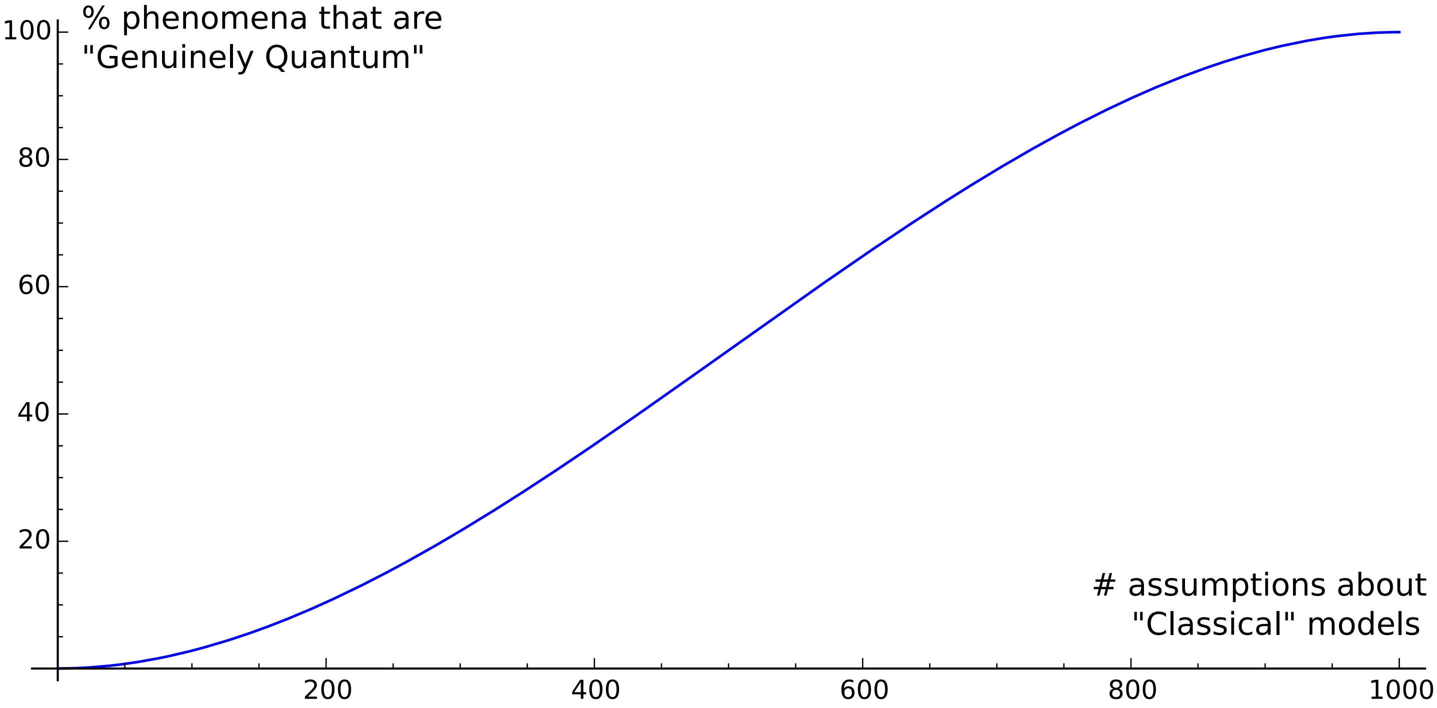
[a] Y. Aharonov and D. Rohrlich, "Quantum Paradoxes" (Wiley, 2005).

The two most meaningless words in physics

“Classical” “Quantum”



Spekkens

- A vast array of seemingly puzzling quantum phenomena occur in classical models with a restriction on how much you can know about the system[1].
- Those that do not, seem to fall under the rubric of After Spekkens (AS) contextuality[2].

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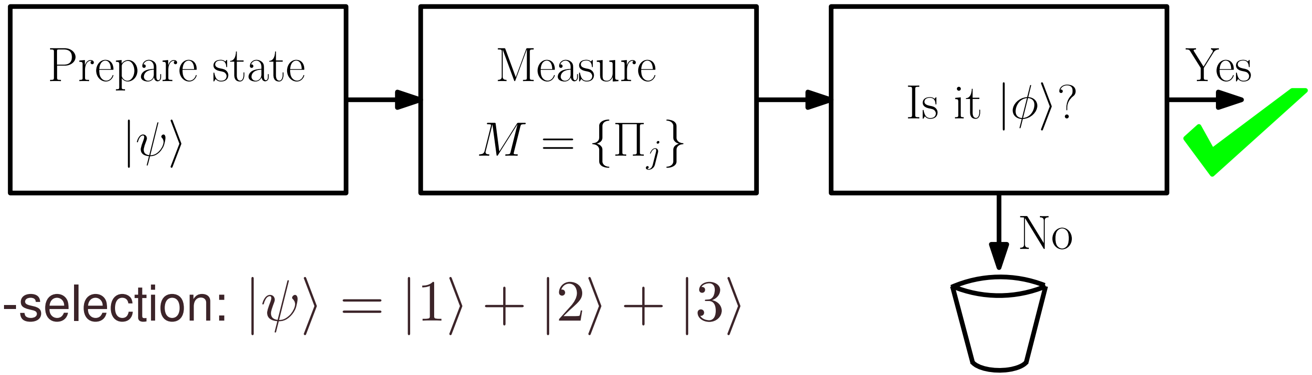
[1] R. Spekkens, *Phys. Rev. A* 75:032110 (2007).

[2] R. Spekkens, *Phys. Rev. A* 71:052108 (2005).

Logical pre- and post-selection paradoxes

Three box paradox

LPPS paradoxes
Three box paradox
BS Contextuality
Non-BS contextual model
AS Contextuality
Discussion and Conclusions



Prepare state $|\psi\rangle$ → Measure $M = \{\Pi_j\}$ → Is it $|\phi\rangle$?

Yes ✓

No

- Pre-selection: $|\psi\rangle = |1\rangle + |2\rangle + |3\rangle$
- Post-selection: $|\phi\rangle = |1\rangle + |2\rangle - |3\rangle$
- Two possible intermediate measurements:
  - ☐ M_1: Is ball in box 1? $\Pi_1 = |1\rangle\langle 1|, \quad \Pi_{2\vee 3} = |2\rangle\langle 2| + |3\rangle\langle 3|$

$$\mathbb{P}(\Pi_1|\psi, M_1, \phi) = 1$$

  - ☐ M_2: Is ball in box 2? $\Pi_2 = |2\rangle\langle 2|, \quad \Pi_{1\vee 3} = |1\rangle\langle 1| + |3\rangle\langle 3|$

$$\mathbb{P}(\Pi_2|\psi, M_2, \phi) = 1$$

Y. Aharonov and L. Vaidman, *J. Phys. A* 24 pp. 2315–2328 (1991).

Before Spekkens Contextuality

Before Spekkens (BS) Noncontextuality

- *Outcome determinism*: At any given time, the system has a definite value for every observable.
  - ☐ For every orthonormal basis $\{|\psi_j\rangle\}$, precisely one of them is asigned the value 1, the rest 0.
- *Noncontextuality*: The outcome assigned to an observable does not depend on which other (commuting) observables it is measured with.
  - ☐ The value assigned to a basis vector does not depend on which basis it occurs in, e.g.

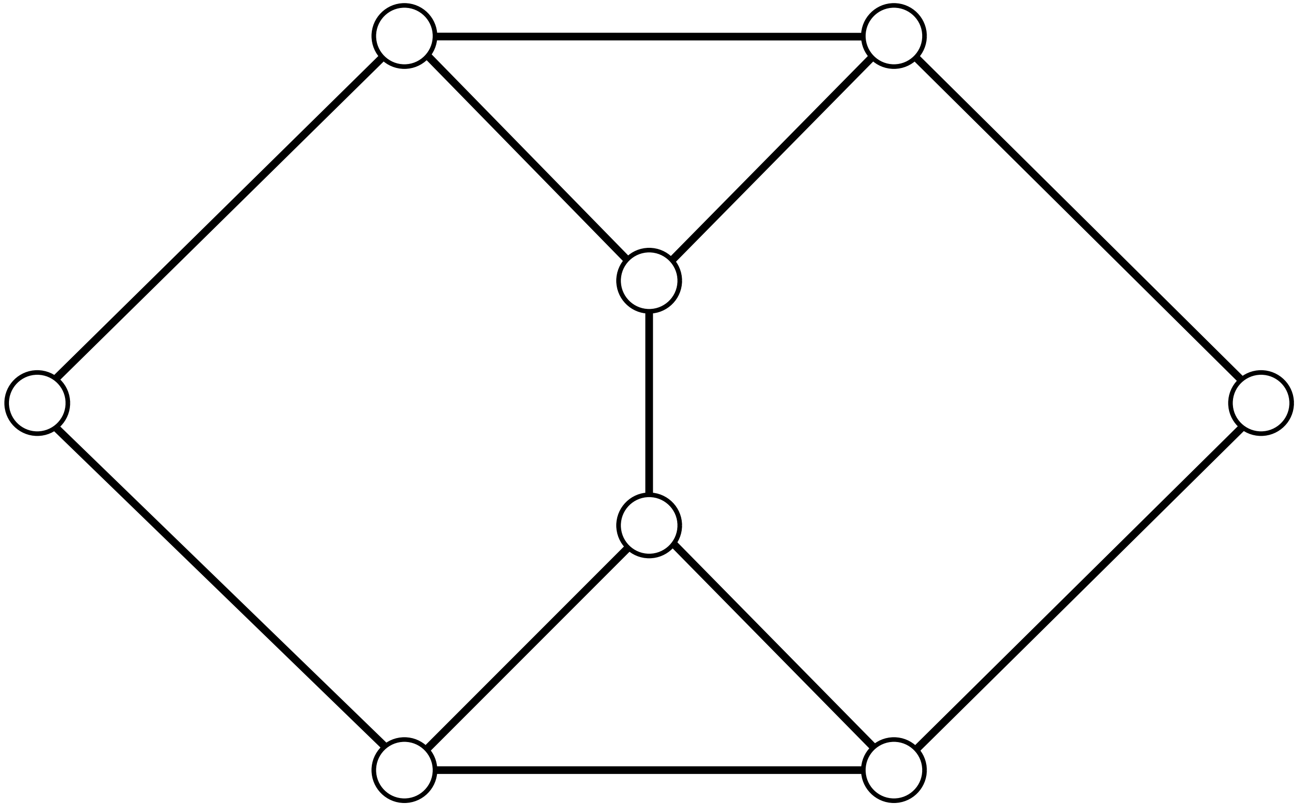
$$|1\rangle\,,|2\rangle\,,|3\rangle$$

vs.

$$|1\rangle\,,|2\rangle+|3\rangle\,,|2\rangle-|3\rangle\,.$$

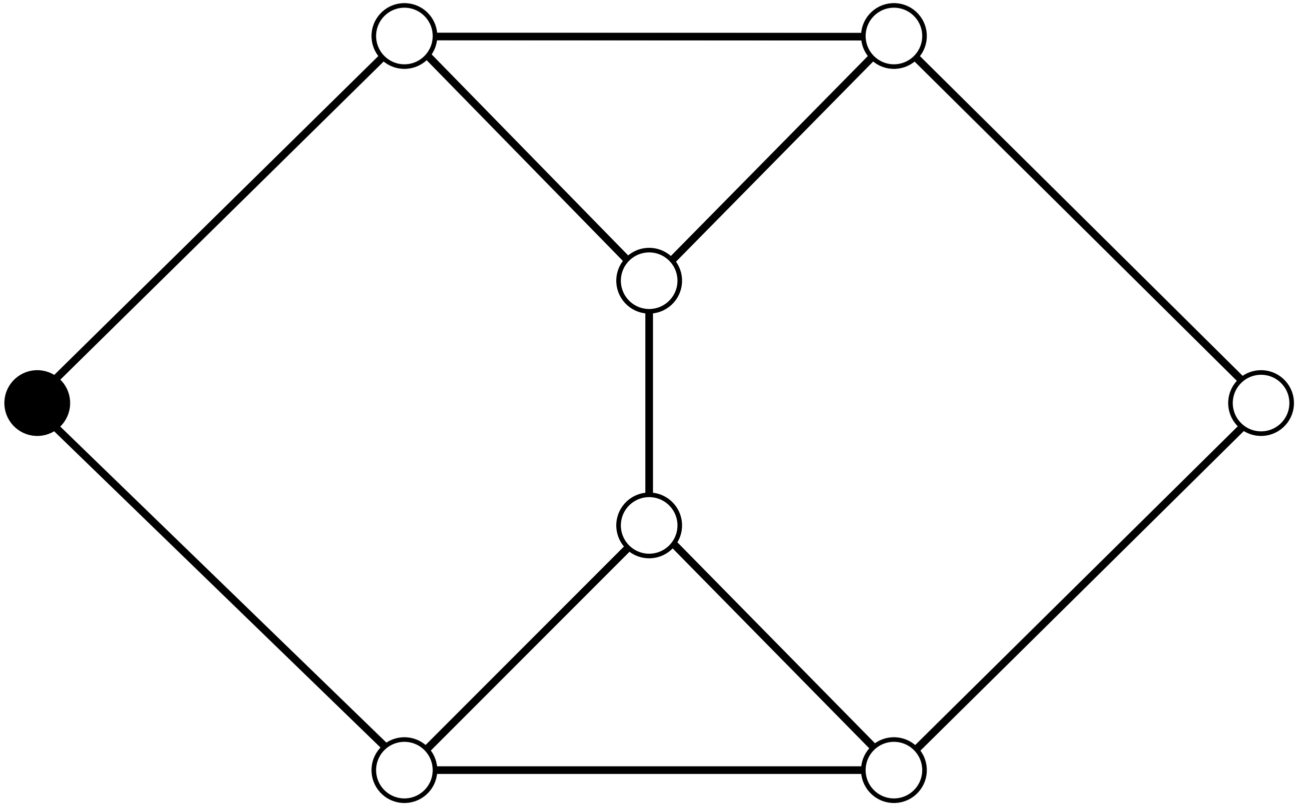
S. Kochen and E. Specker, *J. Math. Mech.* 1 pp. 59–87 (1967).

Clifton’s contextuality proof



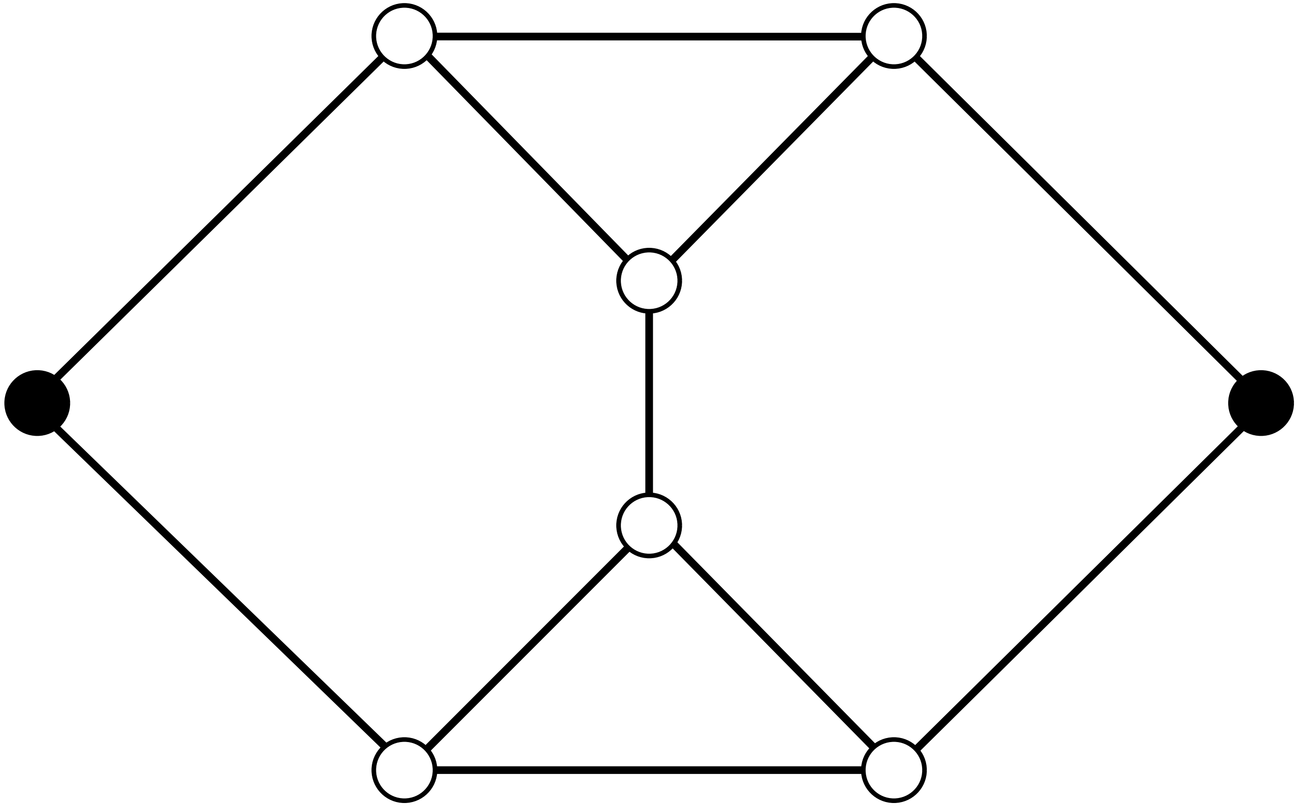
R. Clifton, *Am. J. Phys.* 61 443 (1993).

Clifton’s contextuality proof



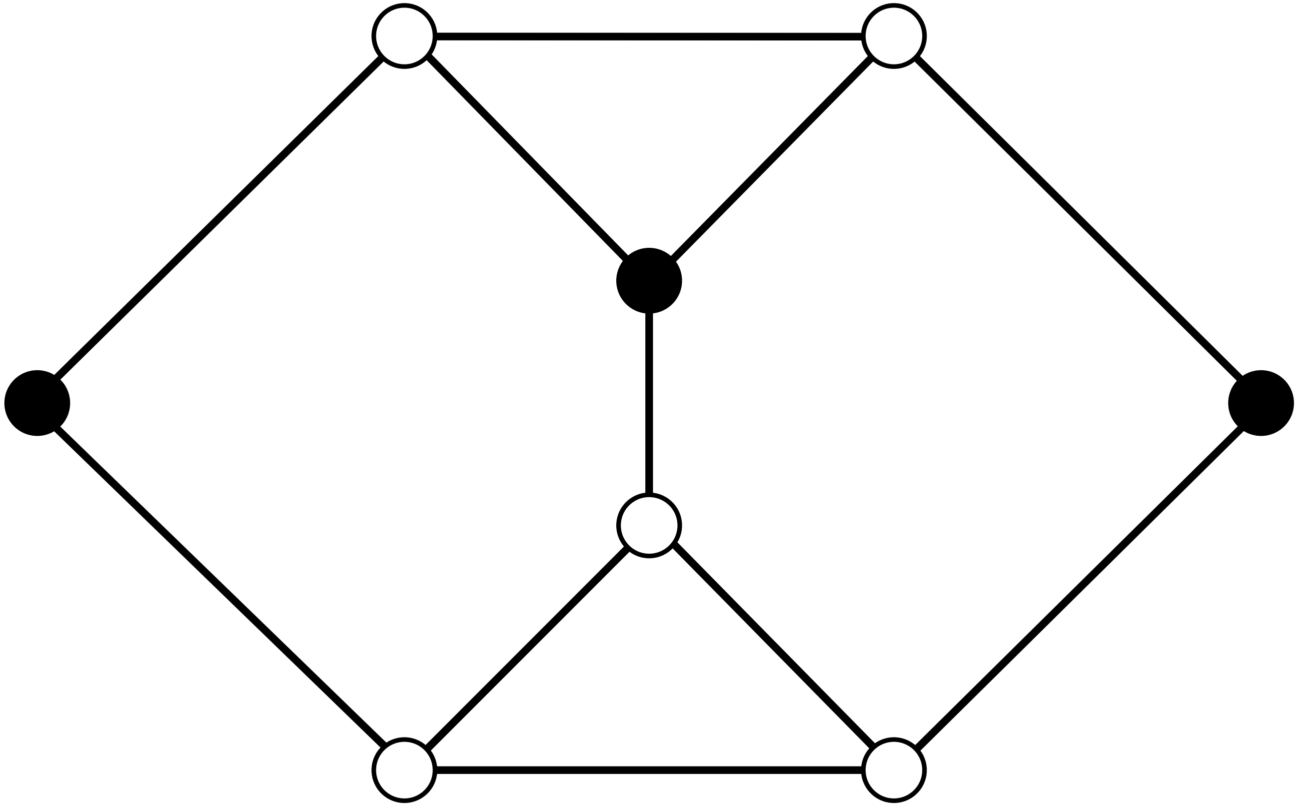
R. Clifton, *Am. J. Phys.* 61 443 (1993).

Clifton’s contextuality proof



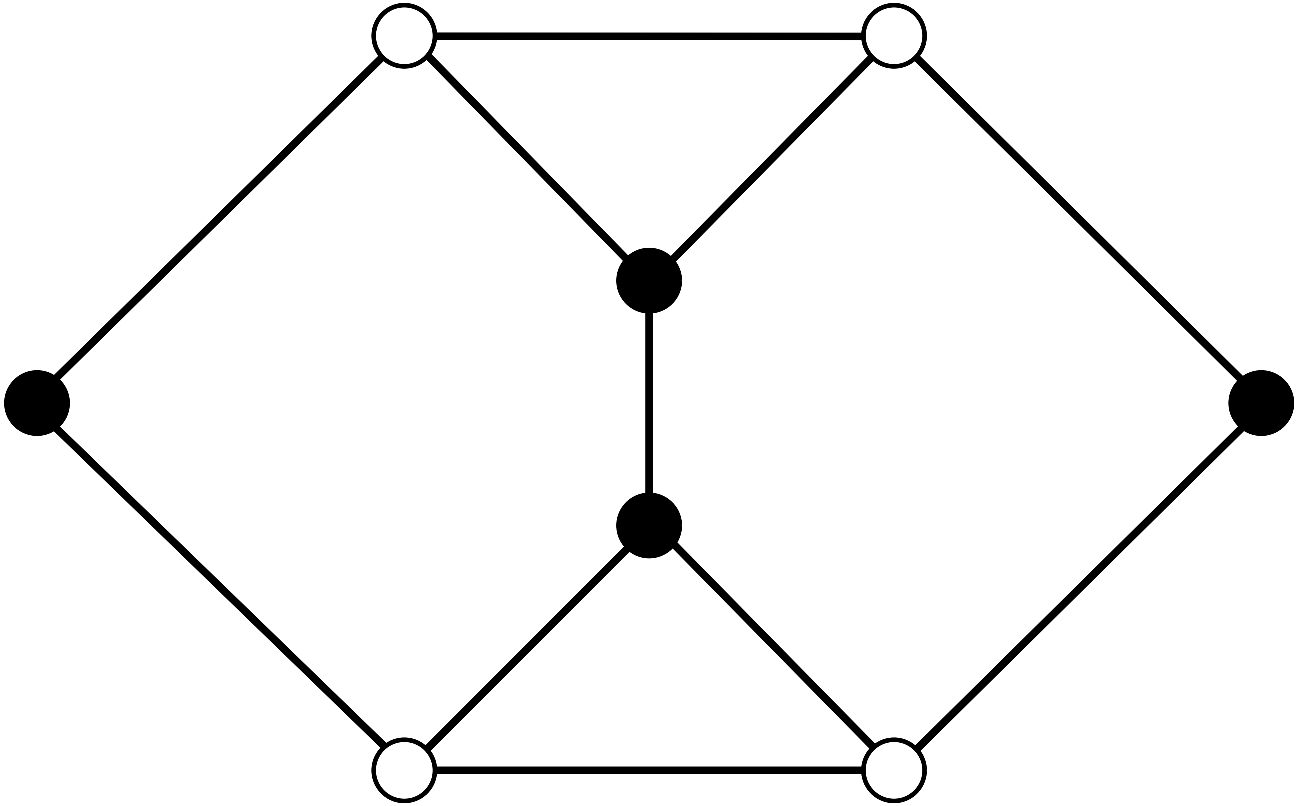
R. Clifton, *Am. J. Phys.* 61 443 (1993).

Clifton’s contextuality proof



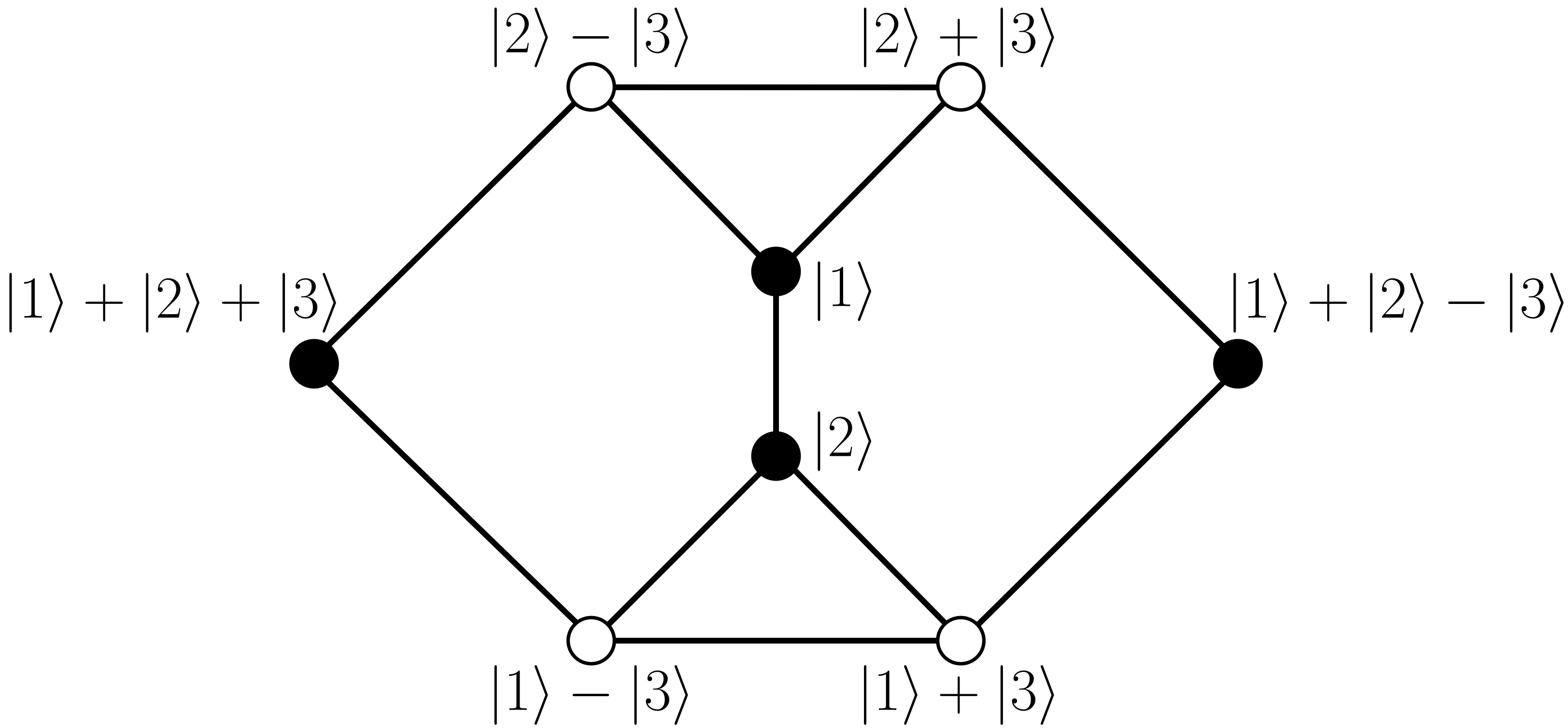
R. Clifton, *Am. J. Phys.* 61 443 (1993).

Clifton’s contextuality proof



R. Clifton, *Am. J. Phys.* 61 443 (1993).

Clifton's contextuality proof



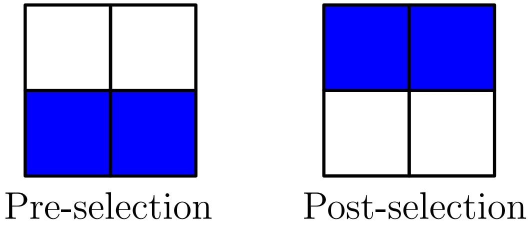
- All logical pre- and post-selection paradoxes are related to a proof of (BS) contextuality in the same way[3].

R. Clifton, *Am. J. Phys.* 61 443 (1993).

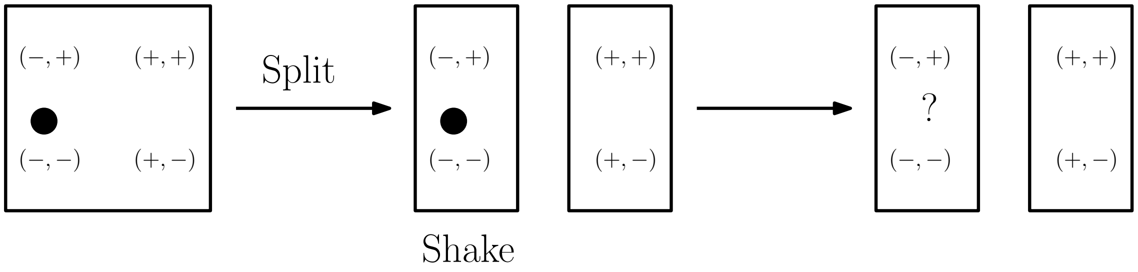
[3] M. Leifer and R. Spekkens, *Phys. Rev. Lett.* 95 200405 (2005).

A non-BS contextual model

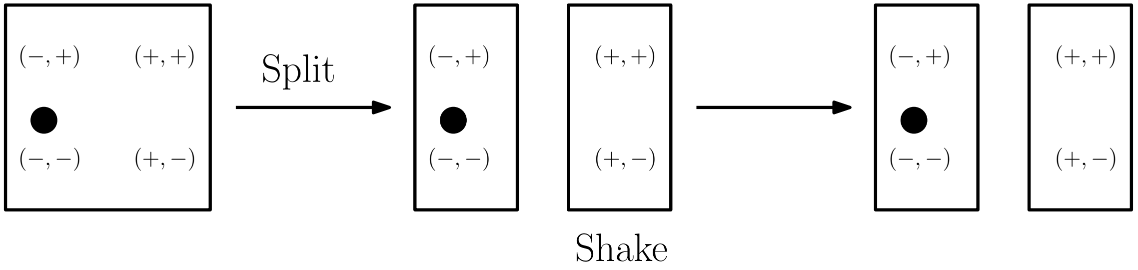
The partitioned box paradox



- “Left”-measurement:

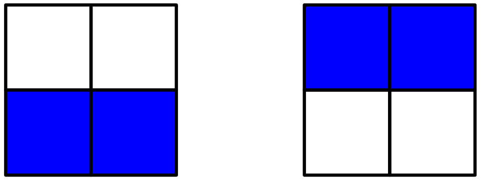


- “Right”-measurement:



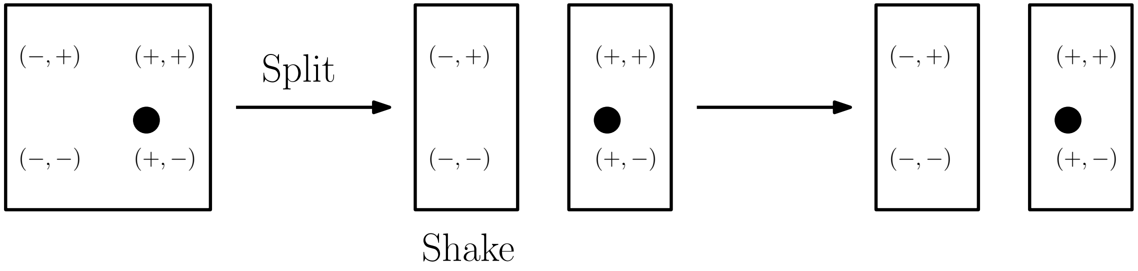
M. Leifer and R. Spekkens, Int. J. Theor. Phys. 44 pp. 1977–1987 (2005).

The partitioned box paradox



Pre-selection Post-selection

- “Left”-measurement:



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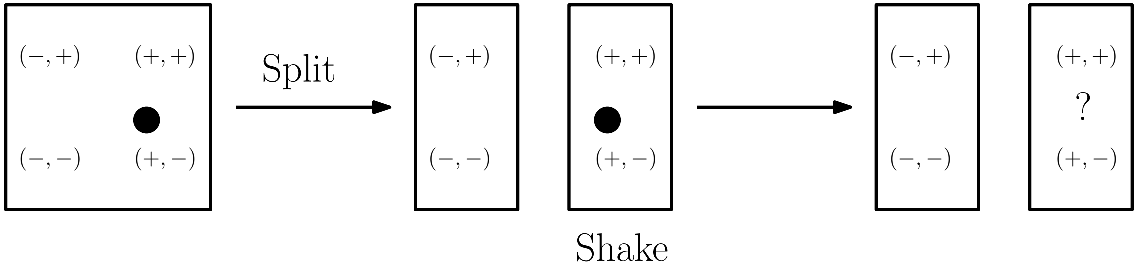
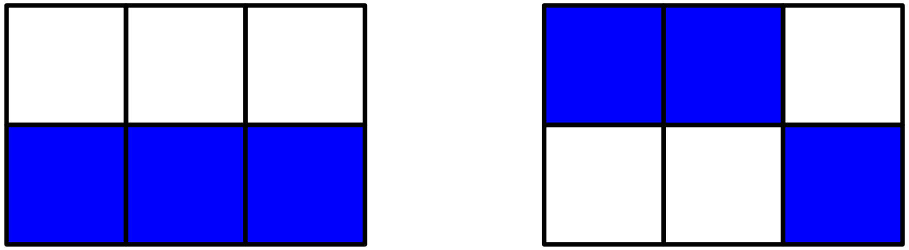


M. Leifer and R. Spekkens, Int. J. Theor. Phys. 44 pp. 1977–1987 (2005).

The partitioned box paradox

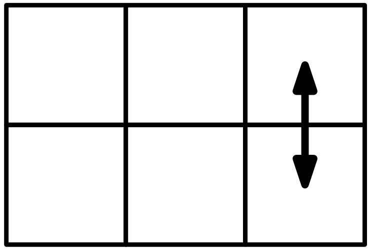
- We can reproduce the predictions of the three-box paradox exactly by adding more states and changing the update rule.

  - ☐ New pre- and post-selection:



Pre-selection   Post-selection

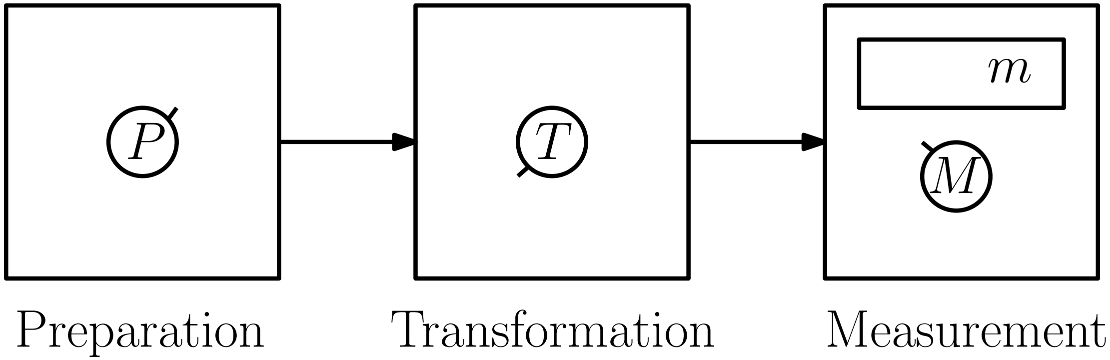
  - ☐ Add this to state-update rule:



After Spekkens Contextuality

After Spekkens (AS) Noncontextuality

- Operational theory:



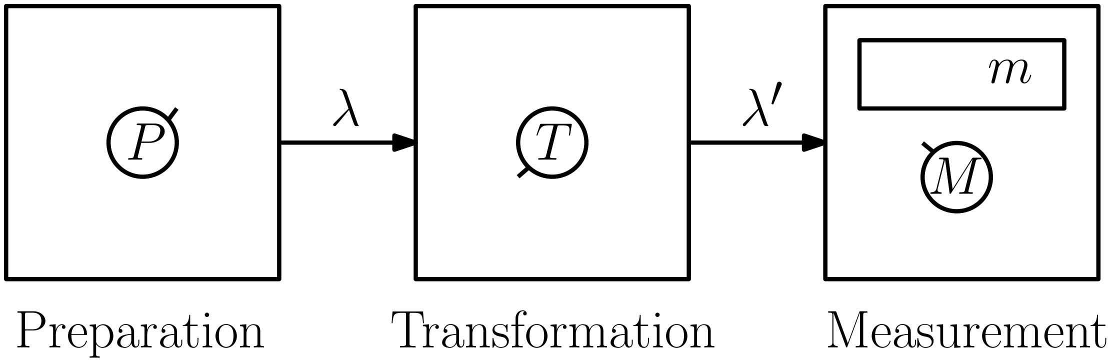
$$\mathbb{P}(m|P,M,T)$$

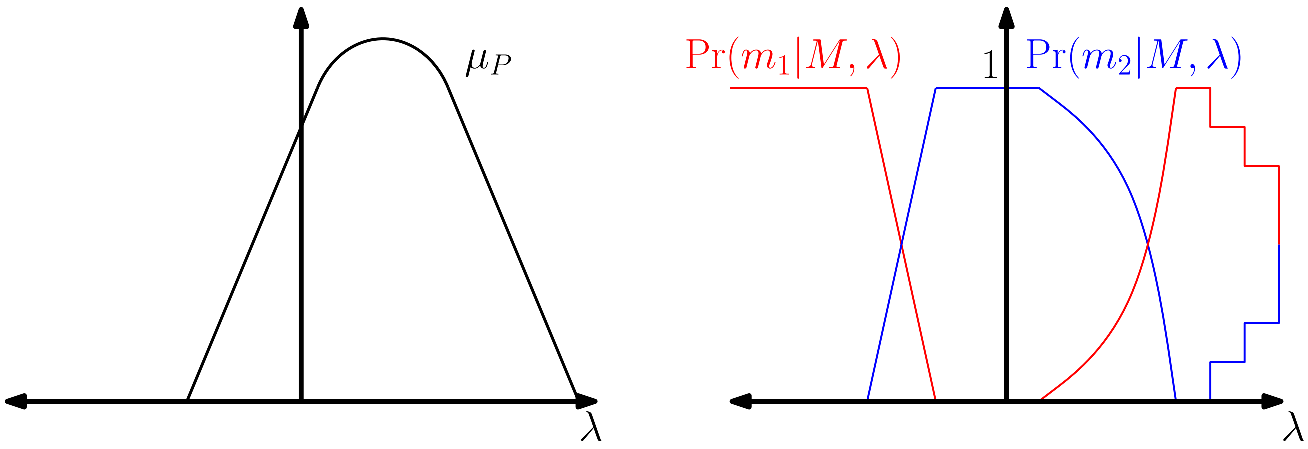
- In quantum theory:

$$\mathbb{P}(m|P,M,T) = \mathrm{Tr}\left(E_m^M \mathcal{E}_T(\rho_P)\right)$$

R. Spekkens, *Phys. Rev. A* 71:052108 (2005).

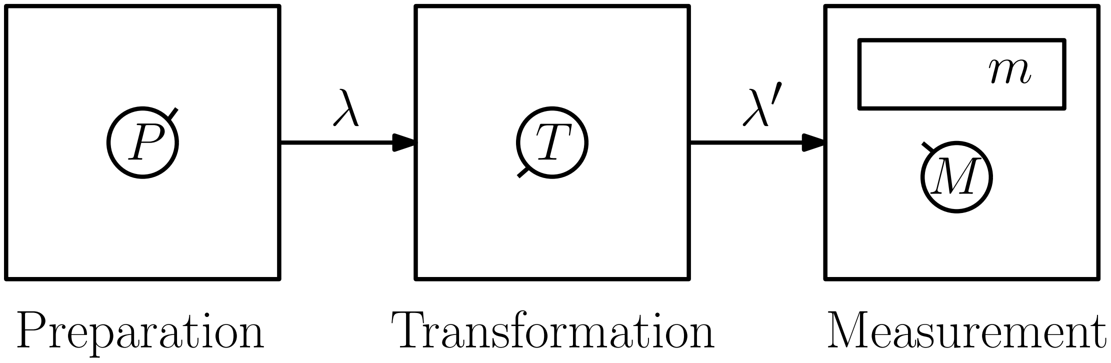
Ontological models





$$\mathbb{P}(m|P,M,T) = \int_{\Lambda'}\int_{\Lambda} \mathrm{Pr}(m|M,\lambda')d\Gamma_T(\lambda'|\lambda)d\mu_P(\lambda)$$

Transformation noncontextuality



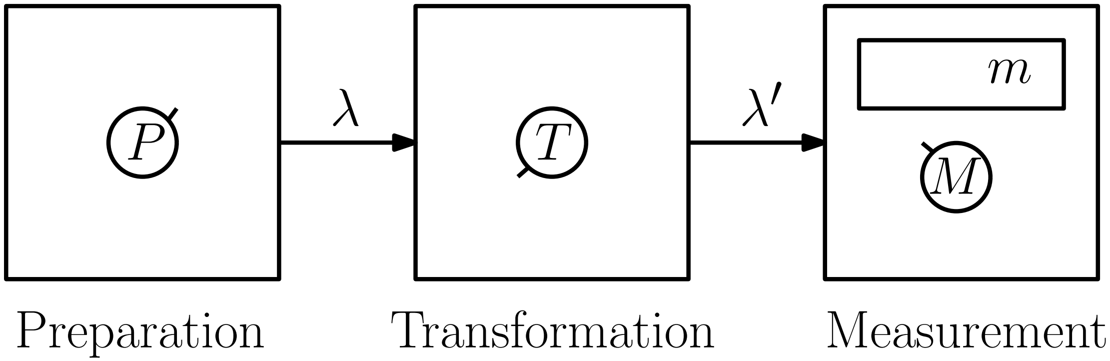
Definition. An ontological model is *transformation noncontextual* if, whenever

$$\mathbb{P}(m|P,M,T) = \mathbb{P}(m|P,M,S)$$

for all P, M, m, we have

$$\Gamma_T = \Gamma_S.$$

Transformation noncontextuality



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- In quantum theory, Γ_T only depends on $\mathcal{E}_T$.

Implications for state-update rules

Theorem. *Let* $\{\Pi_j\}$ *be a projective measurement and let* $\mathcal{E}$ *be the nonselective state-update rule*

$$\mathcal{E}(\rho) = \sum_j \Pi_j \rho \Pi_j .$$

Then,

$$\mathcal{E}(\rho) = p\rho + (1-p)\mathcal{C}(\rho),$$

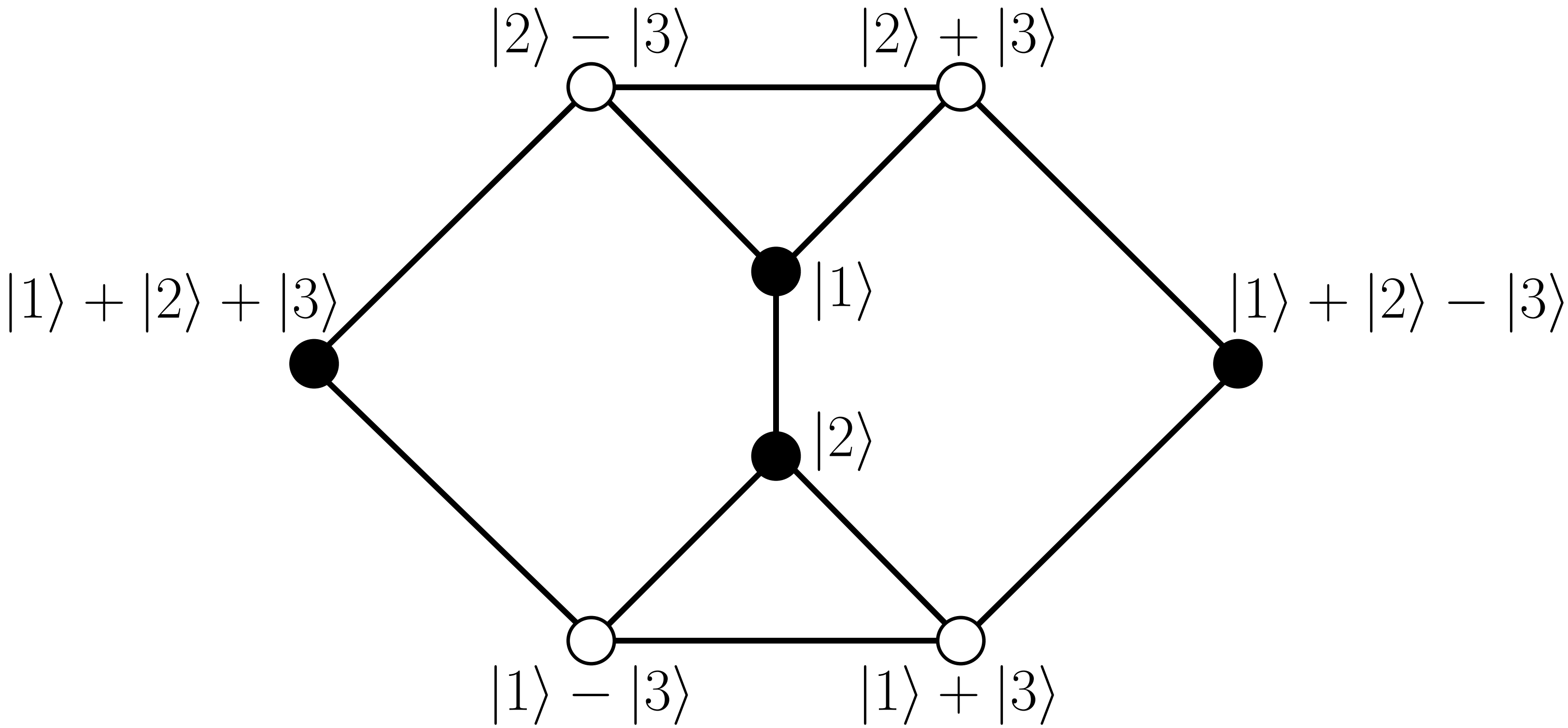
where $\mathcal{C}$ *is a completely-positive, trace-preserving map and* $0 < p \leq 1$*.*

- Proof for special case $\{\Pi_1, \Pi_2\}$:

$$U_1 = \Pi_1 + \Pi_2 = I \qquad\qquad U_2 = \Pi_1 - \Pi_2$$

$$\mathcal{E}(\rho) = \frac{1}{2}U_1\rho U_1^\dagger + \frac{1}{2}U_2\rho U_2^\dagger = \frac{1}{2}\rho + \frac{1}{2}U_2\rho U_2^\dagger .$$

Proof of contextuality



- All logical pre- and post-selection paradoxes are proofs of (PS) contextuality in a similar way.

Discussion and Conclusions

Conclusions

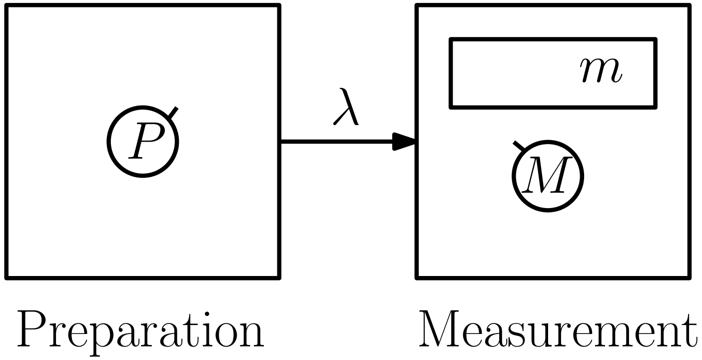
- There is no such thing as a “classical” or “genuinely quantum” phenomenon without
  - ☐ Specifying assumptions for “classical” models.
  - ☐ Specifying which aspects of the phenomenon you want to reproduce.
- A well-motivated set of assumptions is:
  - ☐ Understandable in an AS noncontextual classical probabilistic theory with restriction on knowledge = “classical”.
  - ☐ AS Contextual = “quantum”.
- On this classification LPPS paradoxes are “quantum”.

Weak measurements

- Anomalous weak values have classical analogues:
  - C. Ferrie and J. Combes, *Phys. Rev. Lett.* 113 120404 (2014).
- But, if you try to simulate the quantum predictions exctly, the model must be (AS) contextual:
  - M. Pusey, *Phys. Rev. Lett.* 113 200401 (2014).

Protective measurement & The reality of the quantum state

Preparation noncontextuality



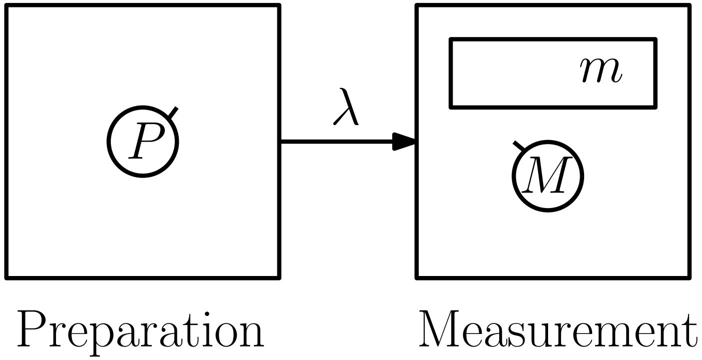
Definition. An ontological model is *preparation noncontextual* if, whenever

$$\mathbb{P}(m|P,M) = \mathbb{P}(m|Q,M)$$

for all M, m, we have

$$\mu_P = \mu_Q.$$

Preparation noncontextuality



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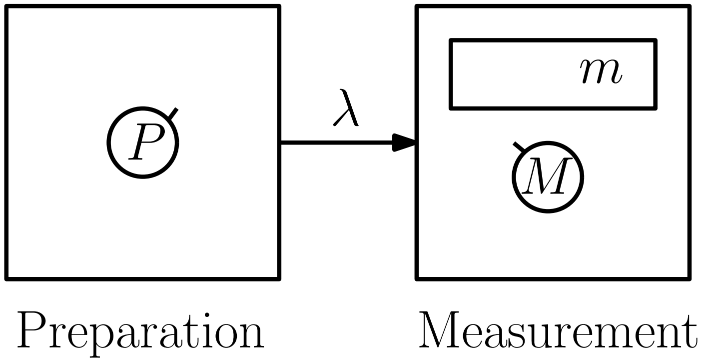
$$\mu_P = \mu_Q.$$

- In quantum theory, μ_P only depends on ρ_P.

Preparation noncontextuality



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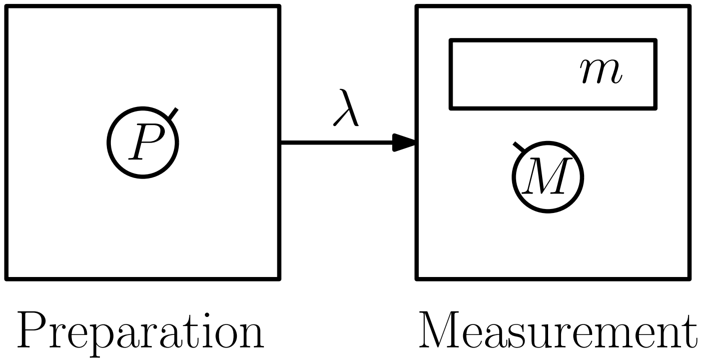
- In quantum theory, μ_P only depends on ρ_P.
- In quantum theory $\Rightarrow$ outcome determinism for projective measurements.

Measurement noncontextuality



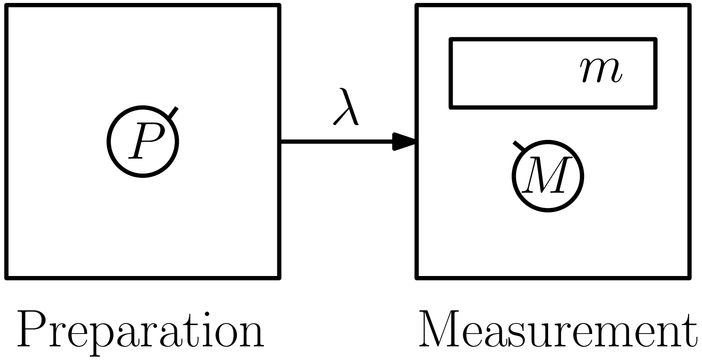
Definition. An ontological model is *measurement noncontextual* if, whenever

$$\mathbb{P}(m|P, M) = \mathbb{P}(n|P, N)$$

for all P, we have

$$\mathrm{Pr}(m|M, \lambda) = \mathrm{Pr}(n|N, \lambda).$$

Measurement noncontextuality



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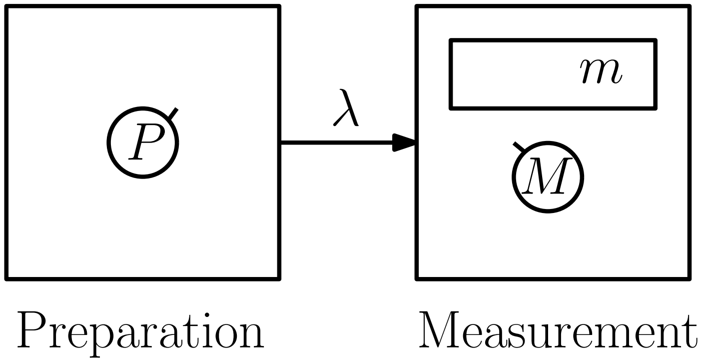
for all P, we have

$$\text{Pr}(m|M,\lambda) = \text{Pr}(n|N,\lambda).$$

- In quantum theory, $\text{Pr}(m|M,\lambda)$ only depends on E_m^M.

Measurement noncontextuality



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for all P, we have

$$\text{Pr}(m|M,\lambda) = \text{Pr}(n|N,\lambda).$$

- In quantum theory, $\text{Pr}(m|M,\lambda)$ only depends on E_m^M.
- In quantum theory, together with preparation noncontextuality, this implies BS noncontextuality.

Collaborators

Josh Combes

Chris Ferrie

Matt Pusey

Protective measurement

- In 1993, Aharonov, Anandan and Vaidman introduced a method of determining the quantum state of a single copy of a quantum system, provided the system is *protected* during the course of measurement[4].

- Protection is a procedure for preventing the quantum state from changing during the course of a measurement. Two types:
  - ☐ Protection via the quantum Zeno effect.
  - ☐ Hamiltonian protection.

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[4] Y. Aharonov, J. Anandan and L. Vaidman, *Phys. Rev. A* 47:6 4616–4626 (1993).

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  - ☐ Protection via the quantum Zeno effect.
  - ☐ Hamiltonian protection.
- Does this imply the reality of the quantum state?

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[6] Y. Aharonov, J. Anandan and L. Vaidman, *Phys. Rev. A* 47:6 4616–4626 (1993).

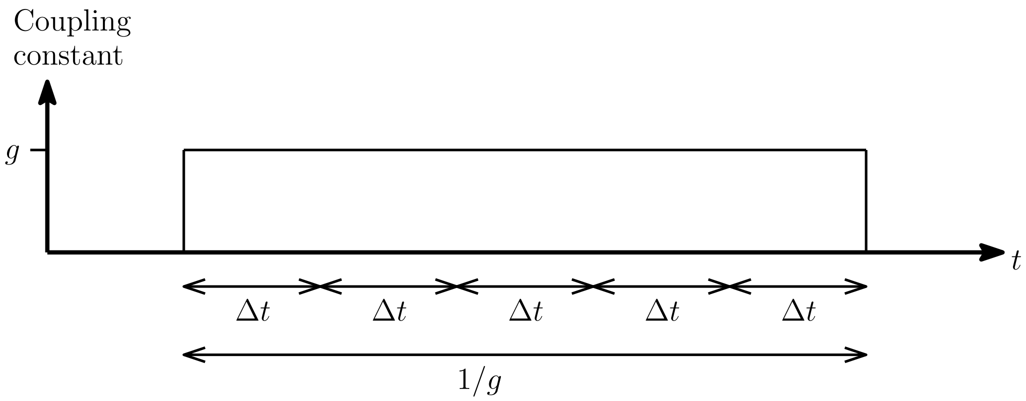
Zeno protected measurement

Alice	Person trying to determine the quantum state
Bob	Person who protects the quantum system.

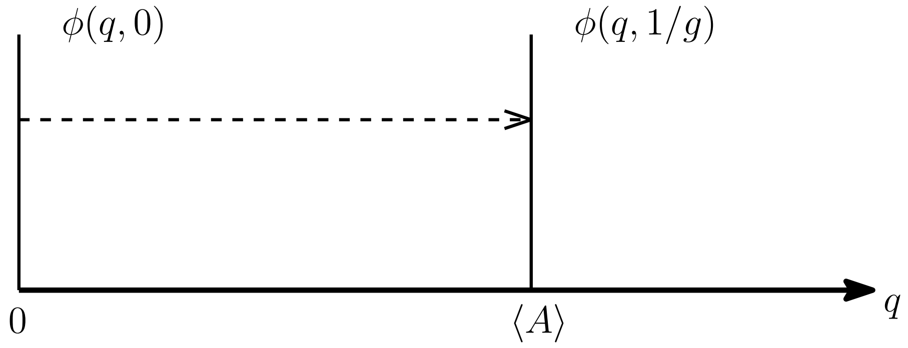
- Bob sends Alice a quantum system prepared in a state $|\psi\rangle$.
- The protection: Every Δt Bob performs a measurement in a basis $\{|\psi_j\rangle\}$ that includes $|\psi\rangle$ as an eigenstate.
- To measure an observable, Alice couples it to a pointer system with wavefunction $\phi(q,t)$ and initial state $\phi(q,0)=\delta(q)$.

Zeno protected measurement

- To measure A, Alice couples the pointer to the system via a Hamiltonian $H = gAp$ for time $1/g$ s.t. $\Delta t \ll 1/g$.



- When $\Delta t \to 0$, the pointer ends up pointing to $\langle A \rangle = \langle \psi | A | \psi \rangle$ and the system remains in state $|\psi\rangle$.

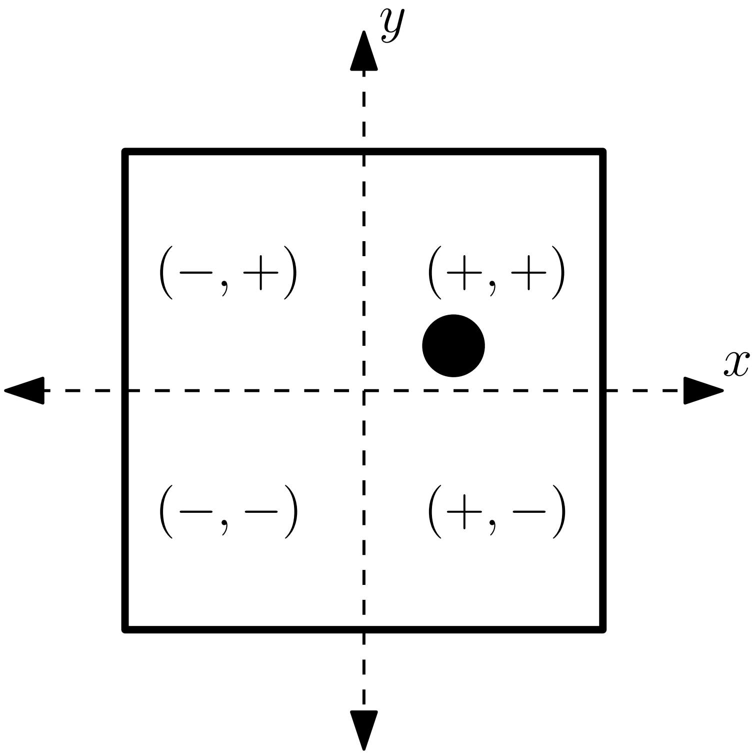


Measuring the quantum state

- Since the state of the system is unchanged, Alice can perform as many protective measurements of different observables as she likes.
- If she measures a tomographically complete set, she can determine the quantum state.
- So does this imply the reality of the quantum state?
- If we can do the same thing with classical probability distributions then the answer is no.

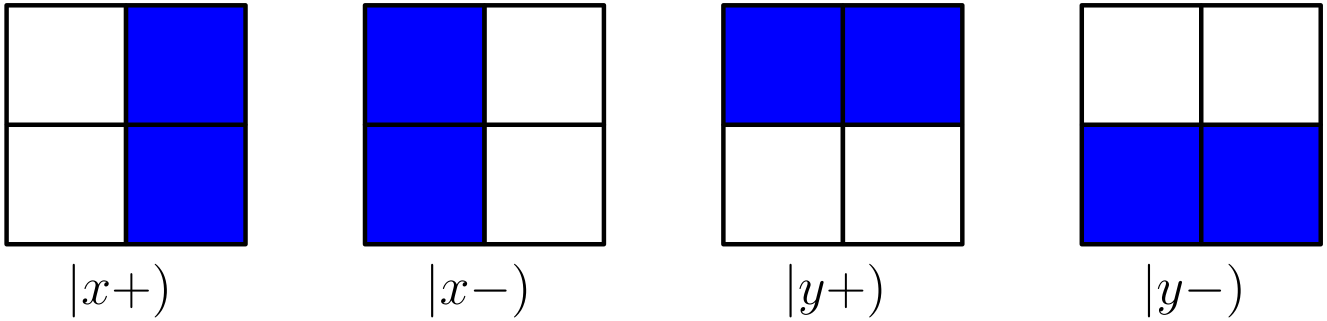
Toy model

- System described by two classical random variables, X and Y, that take values ± 1 (or $\pm$ for short).
  - ☐ (x, y) denotes state in which $X = x$ and $Y = y$.
- Example: Ball in a box:



Toy model: Bob’s States

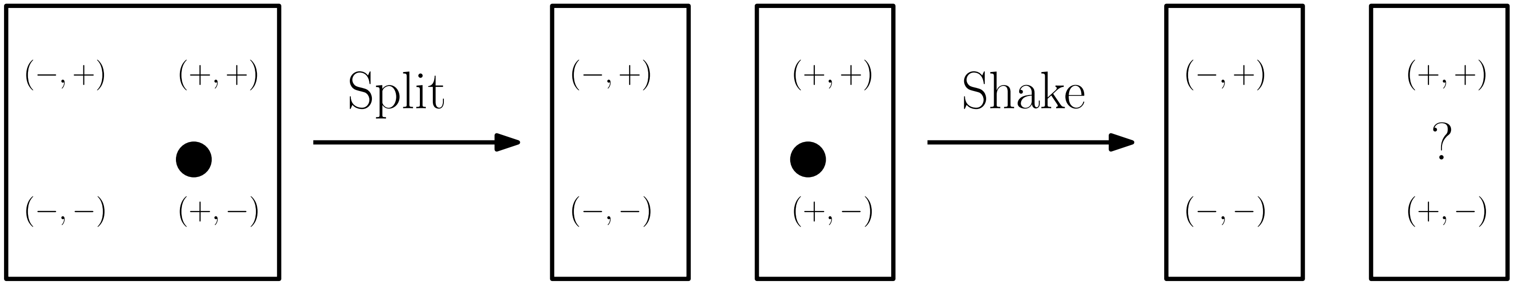
- Assume Bob can prepare the system in four different probability distributions:



Distribution	$\langle X\rangle$	$\langle Y\rangle$
$\|x+)$	+1	0
$\|x-)$	-1	0
$\|y+)$	0	+1
$\|y-)$	0	-1

Toy model: Bob's Measurements

- X-measurement:



$(-,+)$ $(+,+)$ $(-,-)$ $(+,-)$

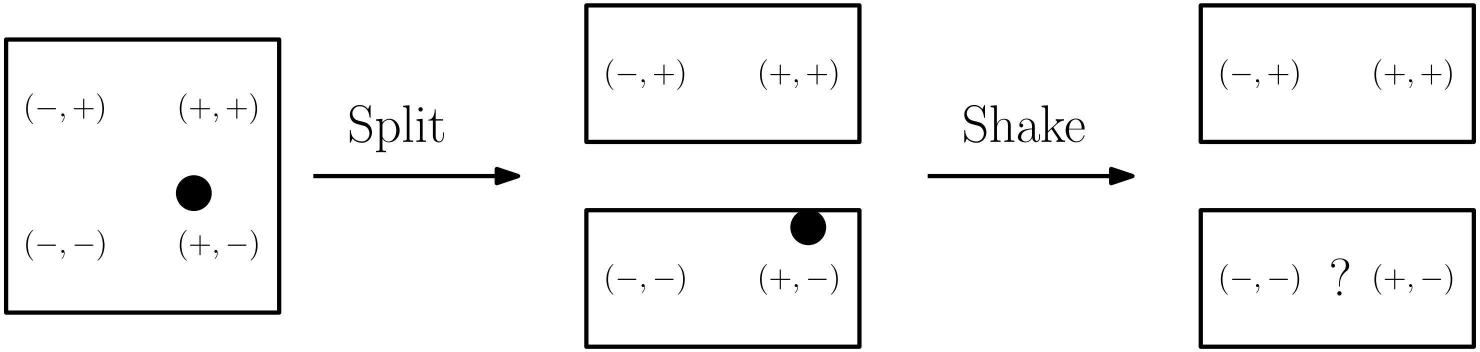
Split

$(-,+)$ $(-,-)$ | $(+,+)$ $(+,-)$

Shake

$(-,+)$ $(-,-)$ | $(+,+)$? $(+,-)$

- Y-measurement:



$(-,+)$ $(+,+)$ $(-,-)$ $(+,-)$

Split

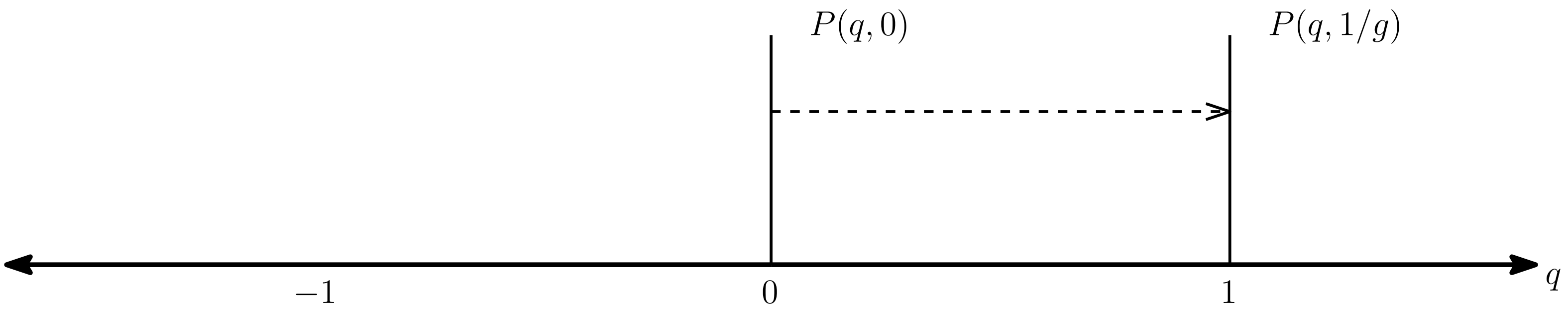
$(-,+)$ $(+,+)$ | $(-,-)$ $(+,-)$

Shake

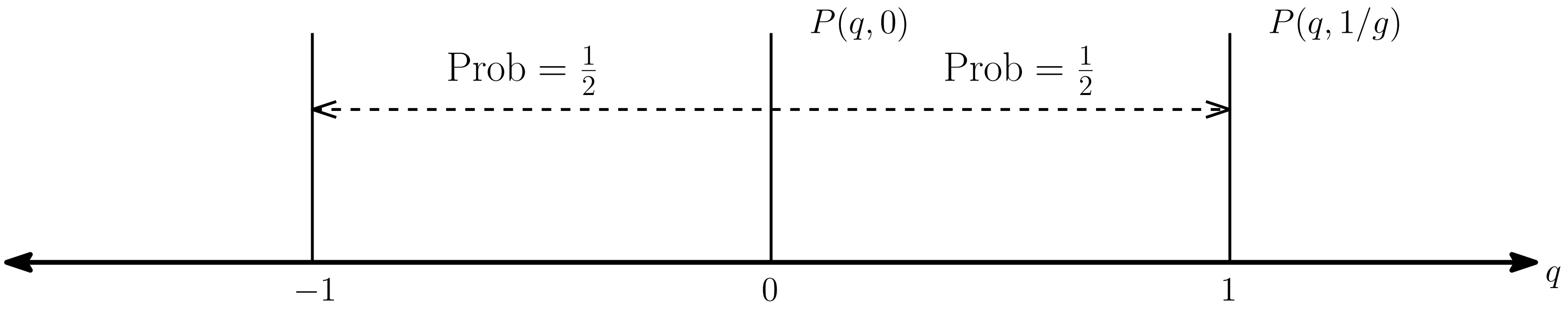
$(-,+)$ $(+,+)$ | $(-,-)$? $(+,-)$

Toy model: Alice's measurements

- System is coupled to a classical pointer prepared in state $q = p = 0$ with Hamiltonian $H = gXp$ or $H = gYp$ for a time $1/g$.
- Without protection, for system prepared in $|x+)$, with $H = gXp$:



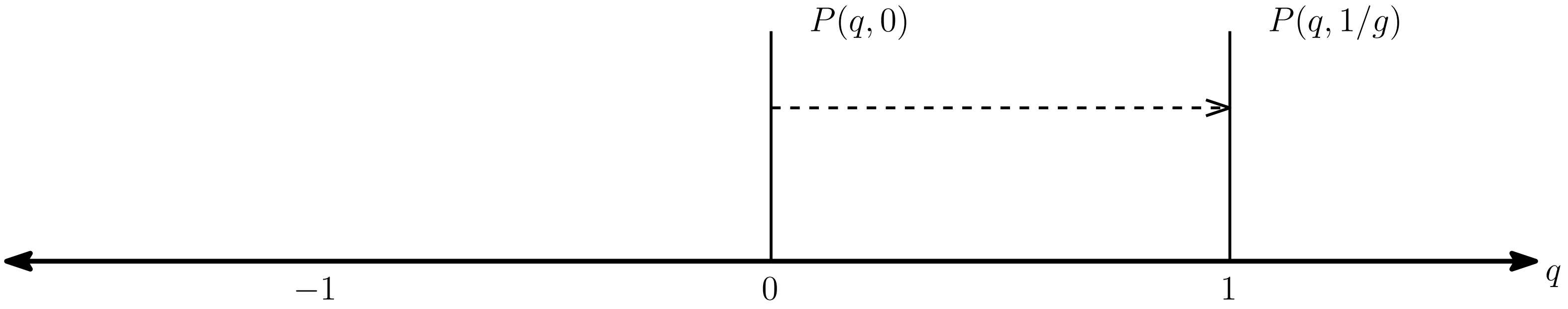
- and with $H = gYp$:



Toy model: Zeno protected measurement

- Now do the same thing whilst at the same time Bob is measuring X every $\Delta t = 1/gN$.

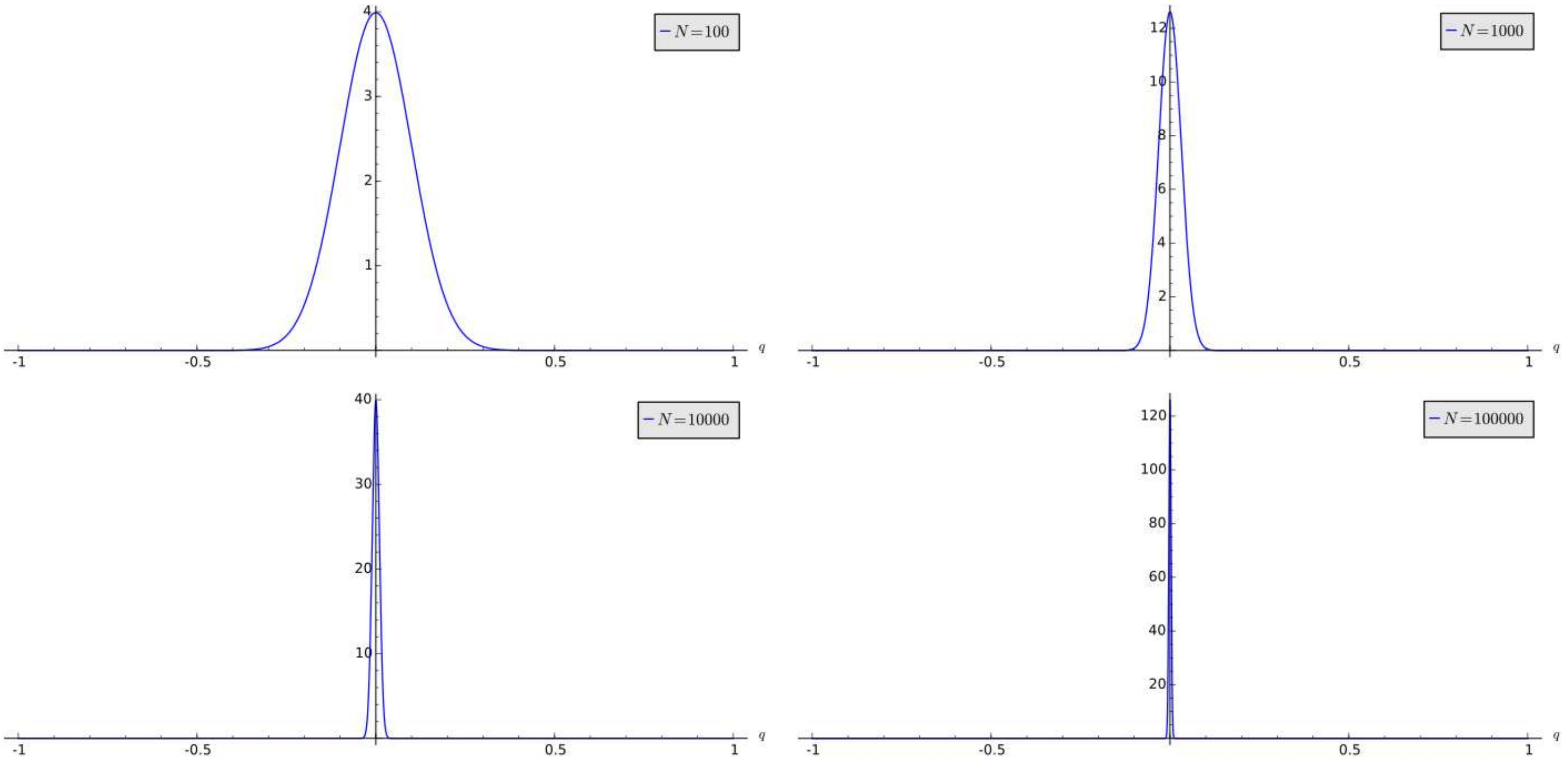
- For $H = gXp$, the pointer moves as before. The pointer is coupled to X, but Bob's measurement only affects Y.



- For $H = gYp$, every Δt the y-coordinate is randomized, so the pointer will keep going in the same direction or switch direction with probability $1/2$ each.
  - Pointer executes an N-step random walk with step size $1/N$.

Toy model: Zeno protected measurement

- For large N, distribution of final pointer position is $\approx \mathcal{N}(0, 1/N)$.
- Tends to $\delta(q)$ as $N \to \infty$.



Comments

- Implicit assumption that if a measurement does not change a quantum state then the measurement does nothing to the system when it is prepared in that state:
  - ☐ Not true in our model: Measuring X randomizes the y-coordinate even though distribution $|x+)$ is unchanged.
- Protective measurement is more like measuring N independently prepared systems than measuring just a single copy.
- One might worry that there are aspects of protective measurement not captured by the toy model.

Exact analysis of protective measurement

- Any sequence of operations on a system that results in a classical outcome can be written in terms of a POVM:

$$\mathbb{P}(q) = \text{Tr}\left(E_q \rho\right)$$

- In a protective measurement E_q is correlated with $|\psi\rangle$ via the protection operation, but it depends only on this and not on the initial state of the system.
- For a protective measurement of a projector, we have shown that

$$E_q = \sum_j |\psi_j\rangle\langle\psi_j| \delta(q - \langle\psi_j| \Pi |\psi_j\rangle)$$

- Thus, most of the information comes from the protection operation.

Further results

- Adding back-action to the Zeno toy model.
- Toy model and exact analysis for Hamiltonian protective measurements.