ψ-EPISTEMIC MODELS ARE EXPONENTIALLY BAD AT EXPLAINING THE DISTINGUISHABILITY OF QUANTUM STATES

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Introduction

The status of the quantum state is one of the most controversial issues in the foundations of quantum theory. Is it an *epistemic* state (state of knowledge or information) or an *ontic* state (state of reality)? In the ontological models framework [1], quantum states are represented by probability measures over a more fundamental set of *ontic states*. An ontological model is deemed *ψ-ontic* if the measures corresponding to distinct states do not overlap and is otherwise *ψ-epistemic*.
Recently, several theorems have been proven that attempt to show that ontic models of must be ψ-ontic [2], but these are based on questionable auxiliary assumptions. Without such assumptions, ψ-epistemic models have been constructed [3], but in these models the amount of overlap between the measures representing nonorthogonal states decreases with Hilbert space dimension, rendering the ψ-epistemic explanation of indistinguishability untenable. In this work, without auxiliary assumptions, I show that the ratio of the overlap of probability measures to the indistinguishability of the quantum states that they represent must always be $\leq de^{-cd}$ for a family of states in $\mathbb{C}^d$, where c is a positive constant and d is divisible by 4 [4]. This improves the previously derived bound of $\leq 4/(d-1)$ for $d \geq 4$ [5].

Ontological Models

Definition: Ontological Model

An *ontological model* for $\mathbb{C}^d$ consists of:

- A measurable space (Λ, Σ) of *ontic states*.
- For each state $|\psi\rangle \in \mathbb{C}^d$, a probability measure $\mu_\psi : \Sigma \to [0,1]$.
- For each orthonormal basis $M = \{|a\rangle, |b\rangle, \ldots\}$, a set of response functions $\xi_a^M : \Lambda \to [0,1]$ satisfying
$$\forall \lambda, \ \sum_{|a\rangle \in M} \xi_a^M(\lambda) = 1.$$

The model is required to reproduce the quantum predictions, i.e.
$$\text{Prob}(a|\psi, M) = \int_\Lambda \xi_a^M(\lambda) d\mu_\psi = |\langle a|\psi\rangle|^2.$$

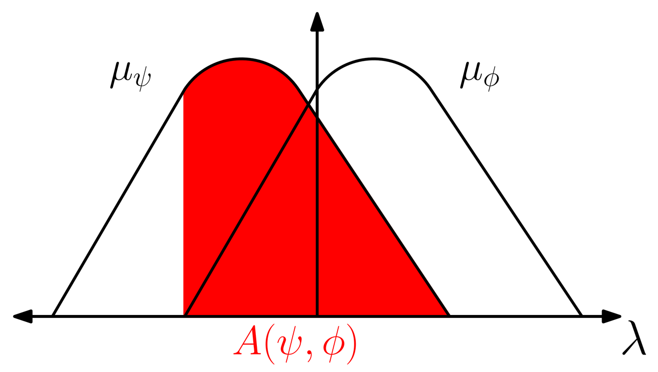
ψ-epistemic explanation of indistinguishability

Suppose that one of two nonorthogonal states, $|\psi\rangle$ and $|\phi\rangle$, is prepared and you do not know which. If the states correspond to overlapping probability measures, then there is a finite probability that the ontic state occupied by the system will end up in the overlap region. If this happens then there is no way of determining which state was prepared with certainty, even if you have access to the full ontic state. As a result, the two states cannot be perfectly distinguished by any procedure, which explains why there is no quantum measurement that does so.
This explanation requires that the amount of overlap of probability measures should be comparable to the indistinguishability of the quantum states they represent, so we need to look at quantitative measures of overlap.

Overlap Measures

Definition: Asymmetric Overlap

The *asymmetric overlap* between two states $|\psi\rangle$ and $|\phi\rangle$ in an ontological model is given by $A(\psi, \phi) := \inf_{\{\Omega \in \Sigma | \mu_\phi(\Omega) = 1\}} \mu_\psi(\Omega)$.



An ontological model is *maximally ψ-epistemic* [6] if $A(\psi, \phi) = |\langle\phi|\psi\rangle|^2$ for all $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$. This is equivalent to
$$\int_\Omega \xi_\phi^M(\lambda) d\mu_\psi = |\langle\phi|\psi\rangle|^2,$$
for all M containing $|\phi\rangle$ and for all Ω such that $\mu_\phi(\Omega) = 1$. This means that the probability of obtaining outcome $|\phi\rangle$ when measuring a system prepared in the state $|\psi\rangle$ is entirely accounted for by the region of overlap between μ_ψ and μ_ϕ.

Definition: Overlap Ratio

The *overlap ratio* between two states $|\psi\rangle$ and $|\phi\rangle$ in an ontological model is
$$k(\psi, \phi) = \frac{A(\psi, \phi)}{|\langle\psi|\phi\rangle|^2}.$$

A maximally ψ-epistemic model has $k(\psi, \phi) = 1$ and a ψ-ontic model has $k(\psi, \phi) = 0$. If $k(\psi, \phi)$ is small then the ψ-epistemic explanation of indistinguishability is untenable, so we are interested in proving upper bounds on $k(\psi, \phi)$.

Graph Theory Definitions

Definition: Orthogonality Graph

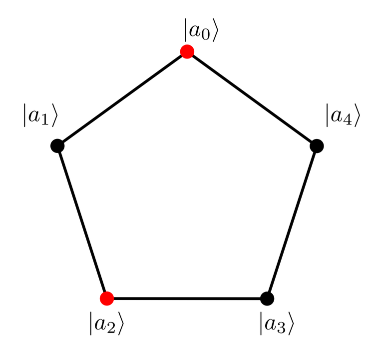
The *orthogonality graph* $G = (V, E)$ of a finite set V of states in $\mathbb{C}^d$ has the states as vertices and there is an edge $(|a\rangle, |b\rangle) \in E$ iff $\langle a|b\rangle = 0$.

Definition: Independence Number

The *independence number* $\alpha(G)$ of a graph $G = (V, E)$ is the largest subset U of V such that no two vertices in U are connected by an edge.

Example: Klyatchko States

The states $|a_j\rangle = \sin\vartheta\cos\varphi_j|0\rangle + \sin\vartheta\sin\varphi_j|1\rangle + \cos\vartheta|2\rangle$ with $0 \leq j \leq 4$, $\varphi_j = \frac{4\pi j}{5}$ and $\cos\vartheta = \frac{1}{\sqrt[4]{5}}$ have a 5-cycle as their orthogonality graph, which has $\alpha(G) = 2$.



Main Result

Theorem

Let V be a finite set of states in $\mathbb{C}^d$ an let $G = (V, E)$ be its orthogonality graph. For any other $|\psi\rangle \in \mathbb{C}^d$ define
$$\bar{k}(\psi) = \frac{1}{|V|} \sum_{|a\rangle \in V} k(\psi, a).$$
Then, in any ontological model
$$\bar{k}(\psi) \leq \frac{\alpha(G)}{|V| \min_{|a\rangle \in V} |\langle a|\psi\rangle|^2}.$$

For the Klyatchko states with $|\psi\rangle = |2\rangle$, this yields $\bar{k}(\psi) \leq \frac{2}{5\frac{1}{\sqrt{5}}} \approx 0.894$, which is a slight improvement over previous results in $\mathbb{C}^3$ [6].

Exponential Bound: Hadamard States

To get the exponential bound in $\mathbb{C}^d$, let
$$|a_{\boldsymbol{x}}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} (-1)^{x_j} |j\rangle,$$
where $\boldsymbol{x} = (x_0, x_1, \ldots, x_{d-1}) \in \{0,1\}^d$ and take $|\psi\rangle = |0\rangle$.
By Frankl-Rödl theorem [7], for d divisible by 4, there exists an $\epsilon > 0$ such that $\alpha(G) \leq (2-\epsilon)^d$. Then,
$$\bar{k}(\psi) \leq \frac{\alpha(G)}{2^d \min_{\boldsymbol{x} \in \{0,1\}^d} |\langle a_{\boldsymbol{x}}|\psi\rangle|^2} = \frac{(2-\epsilon)^d}{2^d \times \frac{1}{d}} = de^{-cd},$$
where $c = \ln 2 - \ln(2-\epsilon)$.

Open Questions

- Error analysis
- Best bounds in small dimensions
- Bounds as a function of inner product

References

[1] N. Harrigan and R. W. Spekkens. *Found. Phys.*, 40(2):125–157, 2010.
[2] M. F. Pusey *et. al.* *Nature Physics*, 8:476–479, 2012. L. Hardy. *Int. J. Mod. Phys. B*, 27:1345012, 2013. R. Colbeck and R. Renner. arXiv:1312.7353. 2013. M. S. Leifer. Is the wavefunction real? a review of ψ-ontology theorems. available at http://mattleifer.info/research/publications/ to appear in Quanta, 2014.
[3] P. G. Lewis *et. al.* *Phys. Rev. Lett.*, 109(15):150404, 2012. S. Aaronson *et. al.* *Phys. Rev. A*, 88(3):032111, 2013.
[4] M. S. Leifer. *Phys. Rev. Lett.*, 112:160404, 2014.
[5] J. Barrett *et. al.* arXiv:1310.8302. 2013.
[6] O. J. E. Maroney. arXiv:1207.6906. 2012.
[7] P. Frankl and V. Rödl. *Trans. Amer. Math. Soc.*, 300(1):259–286, 1987.