A Survey of Quantum Conditional States and Their Applications

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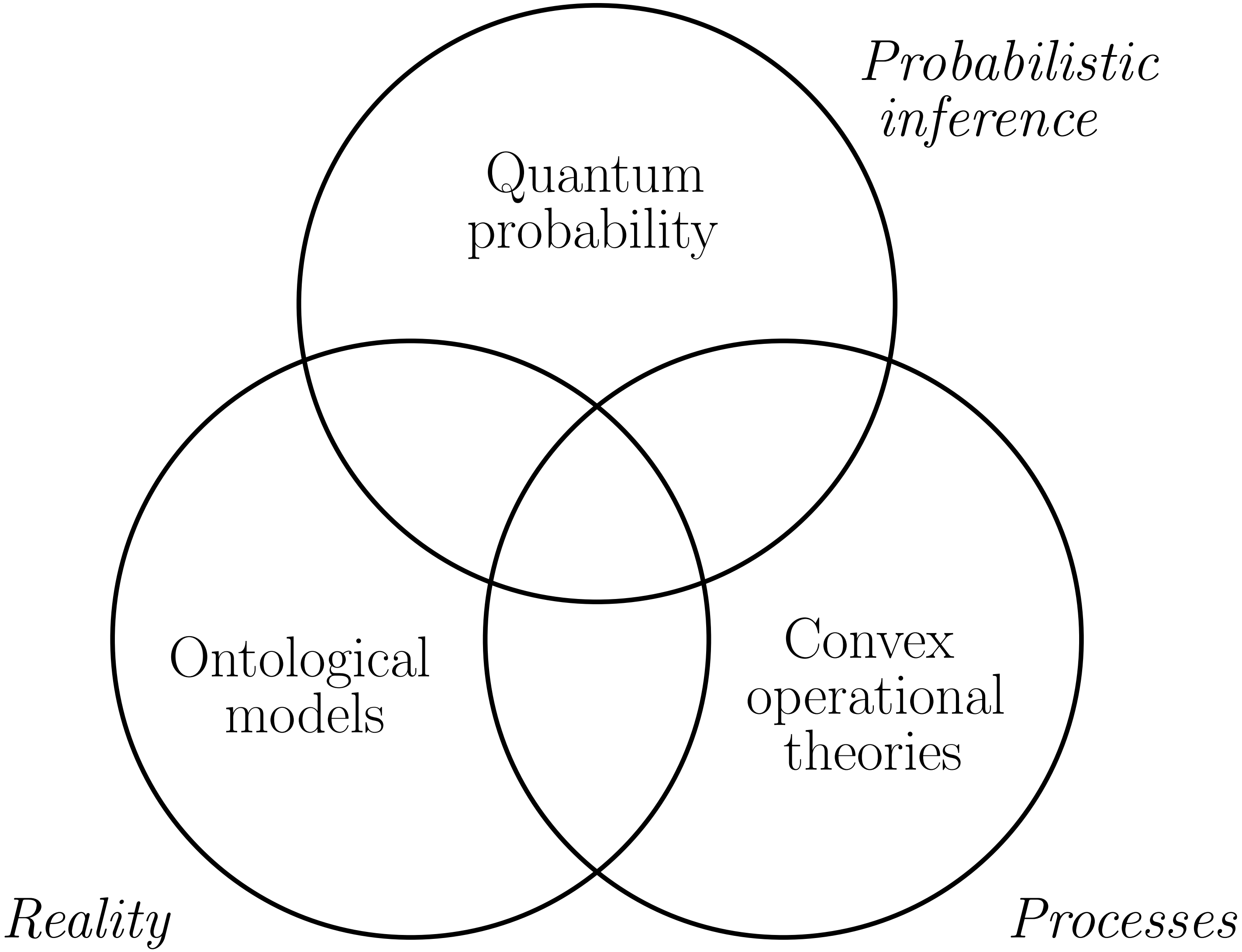
Overview of my research



The goal of quantum probability research

- Goal: Formulate quantum theory as a generalization of abstract probability theory.
  - This involves removing most of the physics, e.g.
    - Background spacetime
    - Kinematics vs. dynamics
    - Starting from physical symmetry representations
- Motivation: To properly understand quantum information and computation a qubit should be an abstract probabailistic object.
- Implications:
  - Novel algorithms and proof methods.
  - Clearer analysis of quantum protocols.
  - Apply quantum theory to anything: Quantum Gravity?

Overview of talk

Introduction

Characterizing Quantum States

Dynamics

Dynamics with intial correlations

Conclusion

Introduction

Textbook quantum theory (finite dimensional version)

- A physical system A is associated with a Hilbert space $\mathcal{H}_A = \mathbb{C}^d$. States of the system are unit vectors $|\psi\rangle \in \mathcal{H}_A$.

- A measurement is associated with a self-adjoint operator $M^\dagger = M$. By the spectral theorem,

$$M = \sum_j m_j \Pi_j.$$

The outcome m_j occurs with probability $\langle\psi| \Pi_j |\psi\rangle$.

- We can alternatively think of a measurement as a set $\{\Pi_j\}$ of orthogonal projection operators with $\sum_j \Pi_j = I_A$.

- A system AB composed of two subsystems A and B is associated with the Hilbert space

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \text{span}\left(|\psi\rangle_A \otimes |\phi\rangle_B\right).$$

Density operators

- More generally, the state of a system A is a positive operator ρ acting on $\mathcal{H}_A$ that satisfies Tr $(\rho) = 1$. The probability of obtaining outcome Π_j in a measurement $\{\Pi_j\}$ is Tr $(\Pi_j \rho)$.

- Examples:
  - *Pure states*: Let $\rho = |\psi\rangle\langle\psi|$. Then,

$$\langle\psi|\,\Pi\,|\psi\rangle = \sum_j \langle\psi|j\rangle\,\langle j|\,\Pi\,|\psi\rangle = \sum_j \langle j|\,\Pi\,|\psi\rangle\,\langle\psi|j\rangle = \text{Tr}\,(\Pi\rho)\,.$$

  - *Mixed states*: If $|\psi_j\rangle$ is prepared with probability p_j then let $\rho = \sum_j p_j\,|\psi_j\rangle\langle\psi_j|$ and then

$$\sum_j p_j\,\langle\psi_j|\,\Pi\,|\psi_j\rangle = \text{Tr}\,(\Pi\rho)\,.$$

Composite systems

- For a joint state ρ_{AB} on $\mathcal{H}_{AB}$, define the reduced state on A as

$$\rho_A = \text{Tr}_B\left(\rho_{AB}\right)$$

where, for an operator,

$$\rho_{AB} = \sum_{jklm} \alpha_{jk;lm} \left|j\right\rangle\left\langle k\right|_A \otimes \left|l\right\rangle\left\langle m\right|_B$$

$$\text{Tr}_B\left(\rho_{AB}\right) = \sum_{jkl} \alpha_{jk;ll} \left|j\right\rangle\left\langle k\right|_A .$$

- Then,

$$\text{Tr}_{AB}\left(\Pi_A \otimes I_B \rho_{AB}\right) = \text{Tr}_A\left(\Pi_A \rho_A\right).$$

Composite systems

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- Then,

$$\mathrm{Tr}_{AB}\left(\Pi_A \rho_{AB}\right) = \mathrm{Tr}_A\left(\Pi_A \rho_A\right).$$

Classical probability as a special case

- Restrict attention to a set of density operators and projectors that commute. Label the eigenbasis $\{|a_j\rangle\}$ with the possible values $\Omega_A = \{a_1, a_2, \ldots\}$ of a classical variable A. Then,

$$\rho = \sum_j P(A = a_j)\,|a_j\rangle\langle a_j| \qquad \Pi_\Lambda = \sum_{a_j \in \Lambda} |a_j\rangle\langle a_j|\,,$$

  where $\Lambda \subseteq \Omega_A$.
- Hence,

$$\text{Tr}\left(\Pi_\Lambda \rho\right) = \sum_{a_j \in \Lambda} P(A = a_j)$$

- For a joint system AB, assume diagonality in a product basis $\{|a_j\rangle_A \otimes |b_k\rangle_B\}$. Then,

$$\rho_{AB} = \sum_{j,k} P(A = a_j, B = b_k)\,|a_j\rangle\langle a_j|_A \otimes |b_k\rangle\langle b_k|_B\,,$$

  and partial trace gives the marginals.

Comparison between classical probability and quantum theory

Classical	**Quantum**
Sample space $\Omega_A = \{a_1, a_2, \ldots\}$	Hilbert space $\mathcal{H}_A = \mathbb{C}^d$
Probability distribution $P(A = a_j) \geq 0$, $\sum_j P(A = a_j) = 1$	Density operator $\rho_A \in \mathfrak{L}^+(\mathcal{H}_A)$, $\text{Tr}_A(\rho_A) = 1$
Cartesian product $\Omega_A \times \Omega_B$	Tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$
Joint distribution $P(A, B)$	Bipartite state ρ_{AB}
Marginal distribution $P(B) = \sum_j P(A = a_j, B)$	Reduced state $\rho_B = \text{Tr}_A(\rho_{AB})$

Conditional probabilities

- Classically, the conditional probability distribution is defined as

$$P(B = b_k | A = a_j) = \frac{P(A = a_j, B = b_k)}{P(A = a_j)}.$$

- What should the quantum analog of this be?
  - ☐ $\rho_{B|A} = \rho_{AB}\rho_A^{-1}$?
  - ☐ $\rho_{B|A} = \rho_A^{-1}\rho_{AB}$?
- Neither of these is positive.

- Define a family of positive products of positive operators

$$G \star^{(n)} H = \left(H^{\frac{1}{2n}} G^{\frac{1}{n}} H^{\frac{1}{2n}} \right)^n .$$

- Two important special cases:
  - $G \odot H = \lim_{n\to\infty} \left(G \star^{(n)} H \right) = e^{(\ln G + \ln H)}$
  - $G \star H = G \star^{(1)} H = H^{\frac{1}{2}} G H^{\frac{1}{2}}$
- Define conditional states:

$$\rho^{(n)}_{B|A} = \rho_{AB} \star^{(n)} \rho_A^{-1} .$$

  - Cerf-Adami: $\rho^{(\infty)}_{B|A} = \rho_{AB} \odot \rho_A^{-1}$
  - The $n = 1$ case: $\rho_{B|A} = \rho_{AB} \star \rho_A^{-1}$

ML, Phys. Rev. A 74 042310 (2006). AIP Conference Proceedings 889 pp. 172–186 (2007).
ML & D. Poulin, Ann. Phys. 323 1899 (2008).
N. Cerf & C. Adami, Phys. Rev. Lett. 79 5194 (1997).

What is special about $\odot$ and $\rho_{B|A}^{(\infty)}$?

- Classical entropy is given by

$$H(A) = -\sum_{A} P(A) \ln P(A),$$

and conditional entropy by

$$H(B|A) = H(A,B) - H(A) = -\sum_{A,B} P(A,B) \ln P(B|A).$$

- Quantum entropy is given by

$$S(A) = -\mathrm{Tr}\left(\rho_A \ln \rho_A\right),$$

and conditional entropy by

$$S(B|A) = S(A,B) - S(A) = -\mathrm{Tr}\left(\rho_{AB} \ln \rho_{B|A}^{(\infty)}\right).$$

N. Cerf & C. Adami, Phys. Rev. Lett. 79 5194 (1997).

- A conditional probability distribution $P(B|A)$ can be defined as a positive function on $\Omega_A \times \Omega_B$ that satisfies

$$\sum_B P(B|A) = 1.$$

- A quantum conditional state $\rho_{B|A}$ with the $\star$-product can be defined as a positive operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ that satisfies

$$\mathrm{Tr}_B\left(\rho_{B|A}\right) = I_A.$$

ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

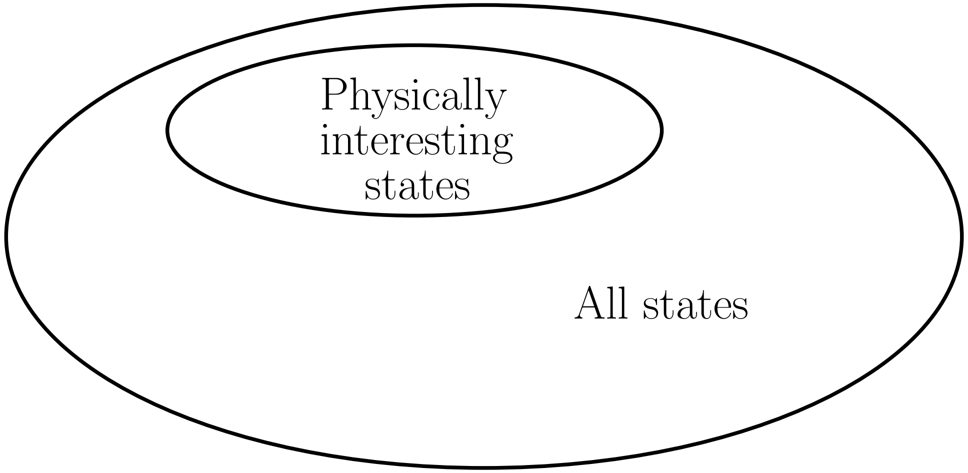
Characterizing Quantum States

Representation

- Generic probability distribution over N variables: $O(d^N)$ params.
- Generic quantum state on N systems: $O(d^{2N})$ params.

Computation of marginals

- $P(A_1) = \sum_{A_2, A_3, \ldots, A_N} P(A_1, A_2, \ldots, A_N)$
- $\rho_{A_1} = \text{Tr}_{A_2 A_3 \ldots A_N} \left(\rho_{A_1 A_2 \ldots A_N}\right)$



Definition. A and B are conditionally independent given C if any of the following equivalent conditions holds:

- $P(A|B,C) = P(A|C)$
- $P(B|A,C) = P(B|C)$
- $P(A,B|C) = P(A|C)P(B|C)$
- $H(A:B|C) = 0$,

where

$$\begin{aligned} H(A:B|C) &= H(A|C) - H(A|B,C) \\ &= H(A,C) + H(B,C) - H(C) - H(A,B,C). \end{aligned}$$

Definition. A and B are conditionally independent given C if $S(A:B|C)=0$, where

$$S(A:B|C)=S(A,C)+S(B,C)-S(C)-S(A,B,C). \quad (1)$$

Theorem. *If* $S(A:B|C)=0$ *then*

- $\rho^{(n)}_{A|BC}=\rho^{(n)}_{A|C}$
- $\rho^{(n)}_{B|AC}=\rho^{(n)}_{B|C}$
- $\rho^{(n)}_{AB|C}=\rho^{(n)}_{A|C}\rho^{(n)}_{B|C}.$

- For $\odot$ all converse implications hold.
- For $\star$ first two converse implications hold.

ML & D. Poulin, Ann. Phys. 323 1899 (2008).

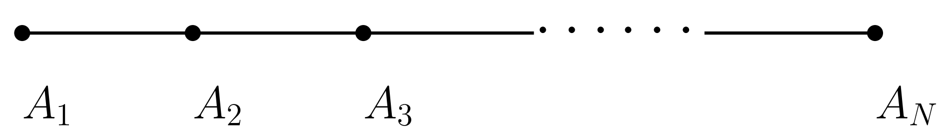
Quantum Markov Chains

- A general state of N systems can be written as

$$\rho_{A_1,A_2,\ldots,A_N} = \rho^{(n)}_{A_N|A_1A_2\ldots A_{N-1}} \star^{(n)} \ldots \star^{(n)} \rho^{(n)}_{A_3|A_2A_1} \star^{(n)} \rho^{(n)}_{A_2|A_1} \star^{(n)} \rho_{A_1}.$$

- Imposing the constraint $S(A_j : A_1A_2\ldots A_{j-2}|A_{j-1}) = 0$ gives

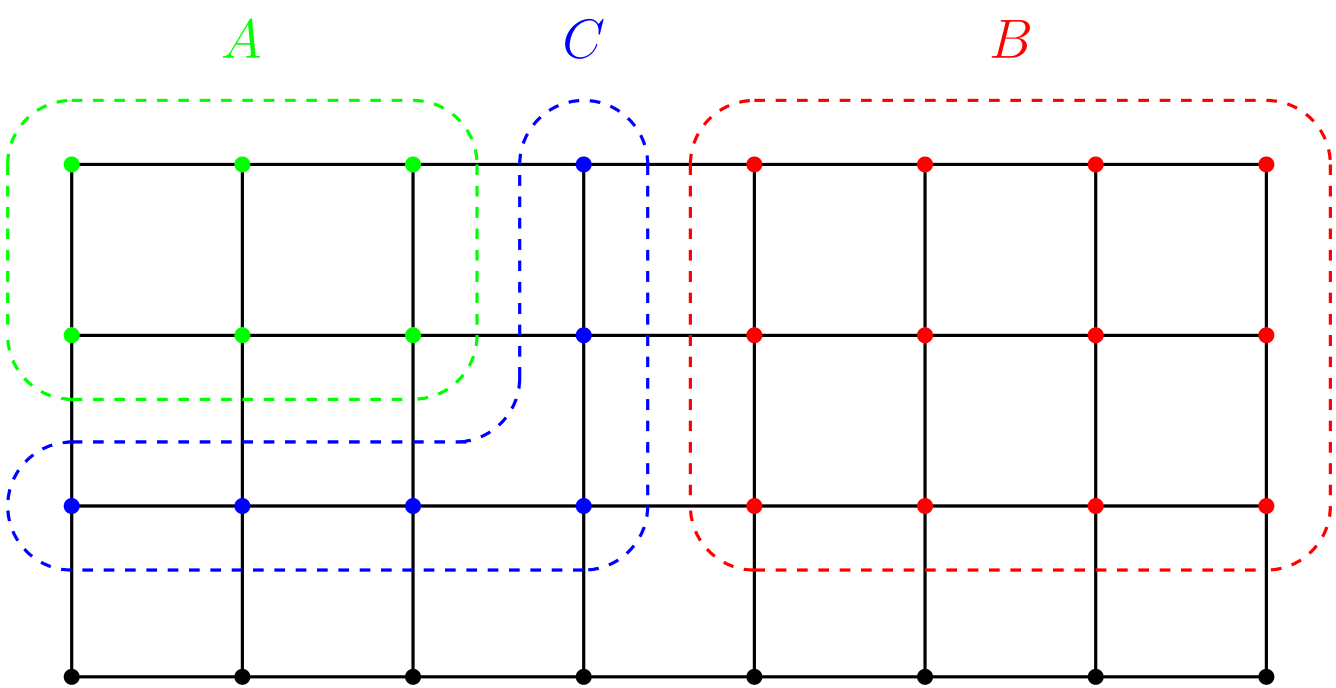
$$\rho_{A_1,A_2,\ldots,A_N} = \rho^{(n)}_{A_N|A_{N-1}} \star^{(n)} \ldots \rho^{(n)}_{A_3|A_2} \star^{(n)} \rho_{A_2|A_1} \star^{(n)} \rho_{A_1}$$



- This decomposition and the one that follows can be used in a quantum generalization of *belief propagation* algorithms.

ML & D. Poulin, Ann. Phys. 323 1899 (2008).

Definition. A *Quantum Markov Network* (G, ρ) is an undirected graph $G = (V, E)$, where the vertices are quantum systems, and a density operator ρ_V that satisfies $S(A : B|C) = 0$ for all disjoint $A, B, C \subseteq V$ such that every path from A to B intersects C.



ML & D. Poulin, Ann. Phys. 323 1899 (2008).

Quantum Hammersley-Clifford Theorem

Theorem. *If (G, ρ) is a Quantum Markov Network and ρ is strictly positive then*

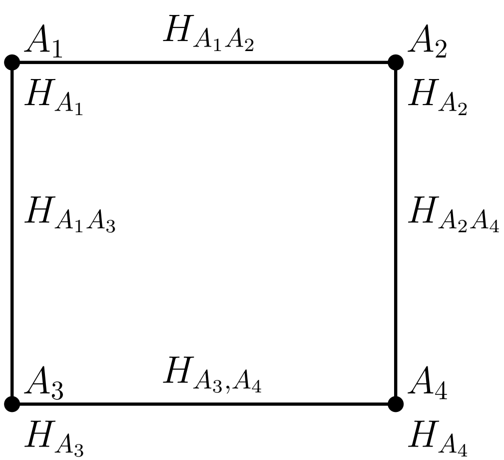
$$\rho_V = \frac{1}{Z} \odot_{C \in \mathfrak{C}} \nu_C,$$

where $\mathfrak{C}$ is the set of cliques in G.

- Alternatively,

$$\rho_V = \frac{1}{Z} e^{-\beta \sum_{C \in \mathfrak{C}} H_C},$$

where $H_C = -\frac{1}{\beta} \ln \nu_C$.



- Converse does not hold: there are extra constraints on the local Hamiltonians.

ML & D. Poulin, Ann. Phys. 323 1899 (2008).

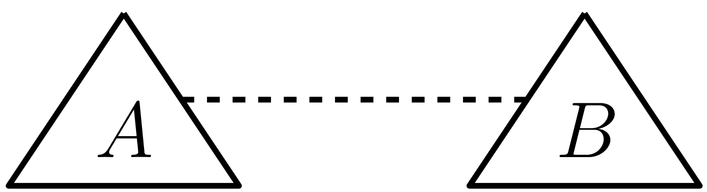
Applications of this work

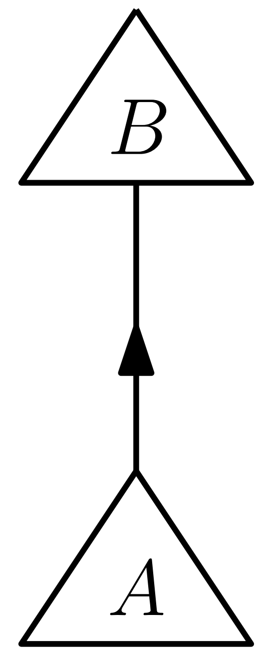
- Applications of Quantum Belief Propagation:
  - Statistical mechanics of quantum Ising spin chains and spin glasses:
    - E. Bilgin and D. Poulin, Phys. Rev. B 81 054106 (2010).
    - C. Laumann, A. Scardicchio and S. L. Sondhi, Phys. Rev. B 78 134424 (2008).
    - D. Nagaj, E. Farhi, J. Goldstone, P. Shor and I. Sylvester, Phys. Rev. B 77 214431 (2008).
  - Study of the connection between the quantum generalization of satisfiability and phase transitions:
    - C. Laumann, R. Moessner, A. Scardicchio and S. L. Sondhi, Quant. Inf. and Comp. vol. 10(1) pp. 1–15 (2010).
- Markov entropy decomposition (dual to belief propagation):
  - Used to obtain lower bounds on the free energy.
    - D. Poulin and M. Hastings, Phys. Rev. Lett. 106 080403 (2011).
    - A. J. Ferris and D. Poulin, Phys. Rev. B 87 205126 (2013).

Dynamics

Classical Probability is Causally Neutral

- Classical probability uses the same formulas for inference regardless of causal structure.





$$P(A,B) = P(B|A)P(A)$$

$$P(B) = \sum_A P(B|A)P(A)$$

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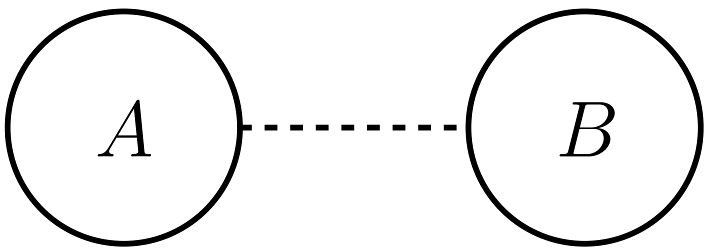
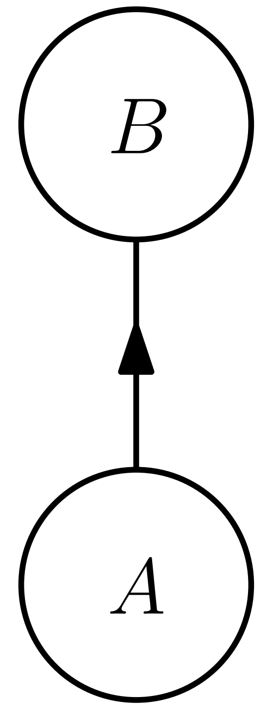
$$P(A,B) = P(B|A)P(A)$$

What about quantum theory?

- By analogy, we would expect:





$$\rho_{AB} = \rho_{B|A} \star \rho_A$$

$$\rho_B = \text{Tr}_A \left(\rho_{B|A}\rho_A\right)$$

$$\rho_B = \text{Tr}_A \left(\rho_{B|A}\rho_A\right)?$$

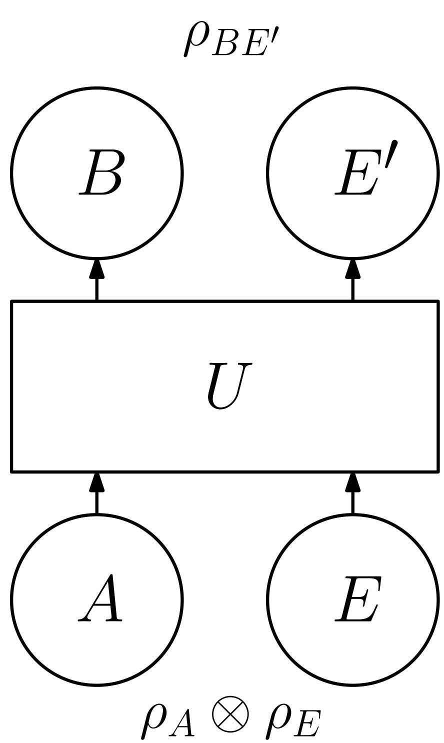
$$\rho_{AB} = \rho_{B|A} \star \rho_A?$$

The usual description of quantum dynamics

- The (discrete time) dynamics of a closed system is given by a the adjoint action of a unitary operator $U^\dagger U = I$.

$$\rho \to U \rho U^\dagger$$

- More generally, dynamics is given by a Completely-Positive Trace-Preserving (CPT) map $\mathcal{E}_{B|A} : \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_B)$.



$$\mathcal{H}_A \otimes \mathcal{H}_E = \mathcal{H}_B \otimes \mathcal{H}_{E'}$$

$$\begin{aligned} \rho_B &= \mathcal{E}_{B|A}(\rho_A) \\ &= \mathrm{Tr}_{E'}\left(U \rho_A \otimes \rho_E U^\dagger\right) \end{aligned}$$

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  - A CP map is a linear map such that

$$\mathcal{E}_{B|A} \otimes \mathcal{I}_E : \mathfrak{L}(\mathcal{H}_A \otimes \mathcal{H}_E) \to \mathfrak{L}(\mathcal{H}_B \otimes \mathcal{H}_E)$$

    is positive for any $\mathcal{H}_E$.
  - A map is trace preserving if

$$\text{Tr}_B \left(\mathcal{E}_{B|A}(M_A)\right) = \text{Tr}_A \left(M_A\right).$$

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- More generally, dynamics is given by a Completely-Positive Trace-Preserving (CPT) map $\mathcal{E}_{B|A} : \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_B)$.
  - A CP map is a map that can be written as

$$\mathcal{E}_{B|A}(\rho_A) = \sum_j M_j \rho_A M_j^\dagger$$

    for some linear operators $M_j : \mathcal{H}_A \to \mathcal{H}_B$.
  - It is trace preserving if

$$\sum_j M_j^\dagger M_j = I_A.$$

The Jamiołkowski isomorphism

- Given a map $\mathcal{E}_{B|A} : \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_B)$, define an operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ via

$$\varrho_{B|A} = \sum_{jk} |j\rangle\langle k|_A \otimes \mathcal{E}_{B|A'}\left(|k\rangle\langle j|_{A'}\right).$$

- Then, $\mathcal{E}_{B|A}(\rho_A) = \mathrm{Tr}_A\left(\varrho_{B|A}\rho_A\right)$.

Theorem. *$\mathcal{E}_{B|A}$ is CPT iff $\varrho_{B|A}^{T_A}$ is a valid conditional state, where, for*

$$\varrho_{B|A} = \sum_{jklm} \alpha_{jk;lm} |j\rangle\langle k|_A \otimes |l\rangle\langle m|_B$$

$$\varrho_{B|A}^{T_A} = \sum_{jklm} \alpha_{kj;lm} |j\rangle\langle k|_A \otimes |l\rangle\langle m|_B .$$

A. Jamiołkowski, Rep. Math. Phys. 3 pp. 275–278 (1972)

ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

Acausal and causal conditional states

Definition. An *Acausal Conditional State* of B given A is a positive operator $\rho_{B|A}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$\text{Tr}_B\left(\rho_{B|A}\right) = I_A.$$

Definition. A *Causal conditional state* of B given A is an operator $\varrho_{B|A}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that $\varrho_{B|A}^{T_A}$ is an acausal conditional state.

Definition. A *Causal joint state* on AB is an operator ϱ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ that can be written as

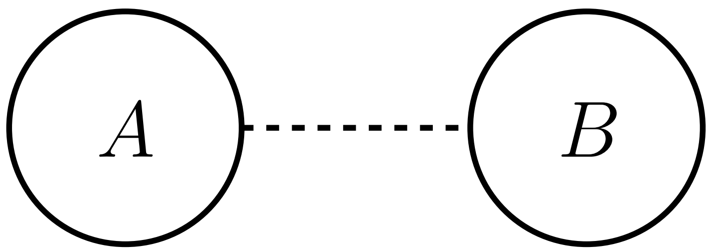
$$\varrho_{AB} = \varrho_{B|A} \star \rho_A$$

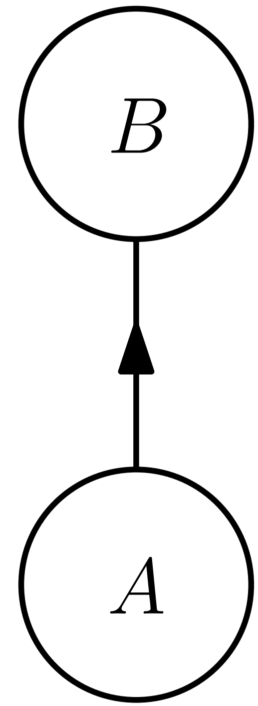
for some causal conditional state $\varrho_{B|A}$ and marginal state ρ_A.

Alternatively $\varrho_{AB}^{T_A}$ is an acausal joint state.

ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

Rewriting dynamics





$$\rho_{AB} = \rho_{B|A} \star \rho_A$$

$$\rho_B = \mathrm{Tr}_A \left(\rho_{B|A}\rho_A\right)$$

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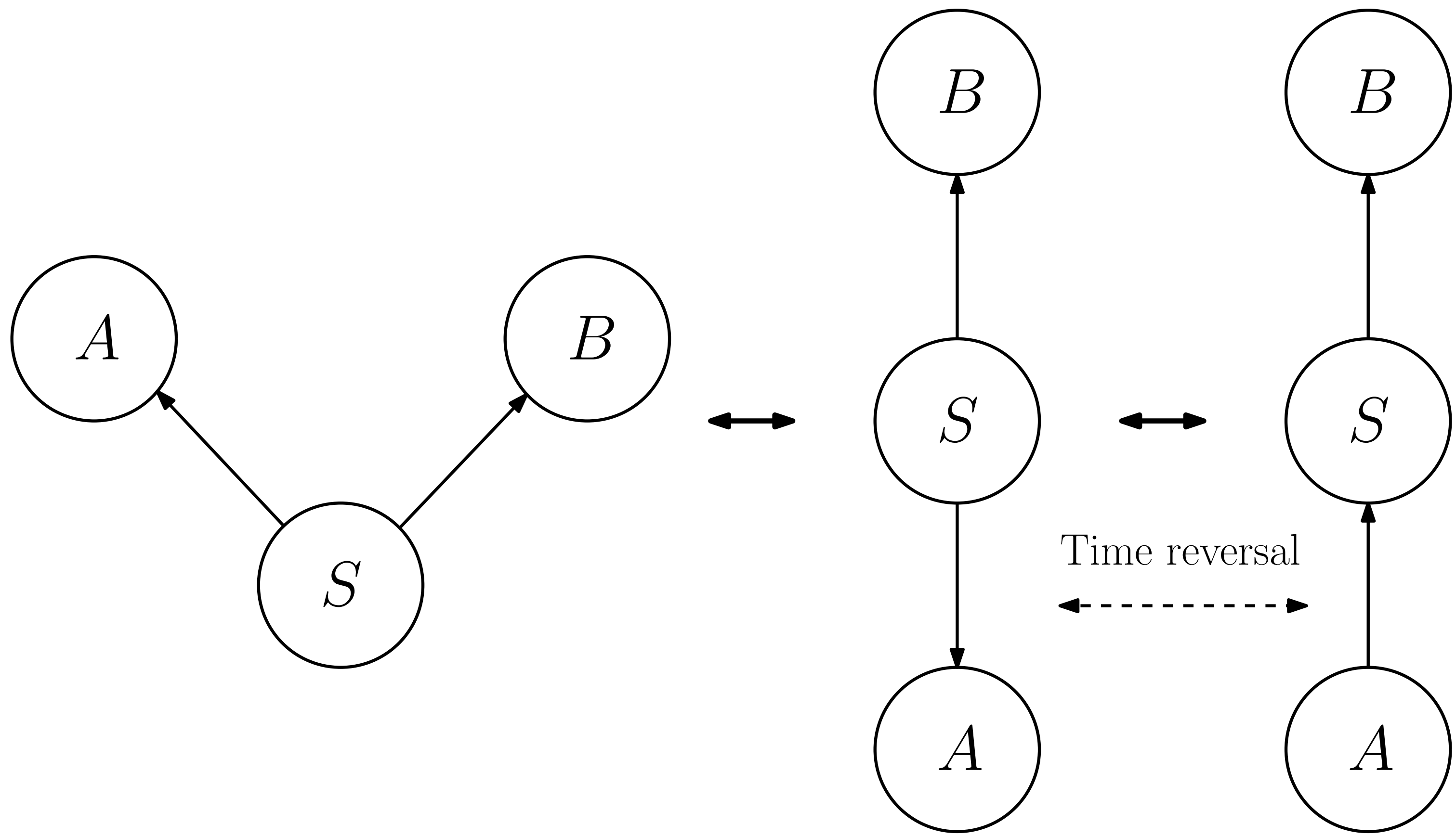
$$\varrho_{AB} = \varrho_{B|A} \star \rho_A$$

ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

Why is there a partial transpose?



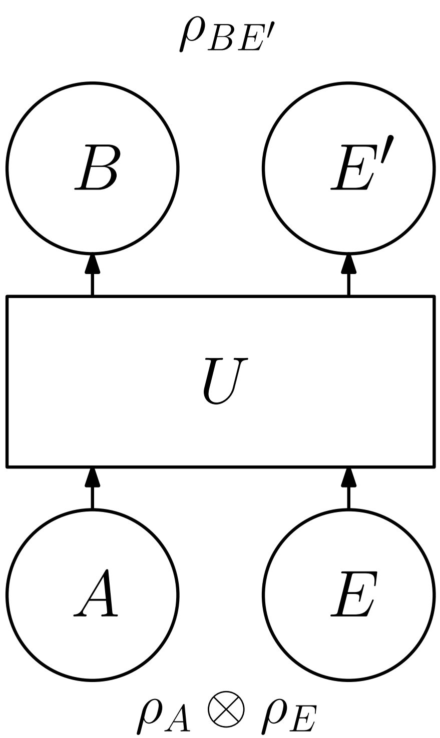
Further work and applications

- ML and R. Spekkens, Phys. Rev. A 88 052130 (2013).
  - Unified formalism for preparations, measurements and dynamics
  - Quantum Bayes theorem
  - Retrodictive quantum theory
  - Quantum steering
- ML and R. Spekkens, to appear in J. Phys. A (2014).
  - Quantum sufficient statistics
  - Quantum state compatibility
  - Quantum state improvement and pooling
- B. Coecke & R. Spekkens, Synthese 186 651 (2012).
  - Category theoretic version of quantum Bayesian inference.
- E. G. Cavalcanti & R. Lal (2013). arXiv:1311.6852.
  - Used to analyse quantum generalization of Bell's locality condition.
- J. Norton (2014). `http://bit.ly/1km1Q4L`.
  - Quantum inductive logic

Dynamics with intial correlations

Dynamics with initial correlations

- Recall, in deriving CPT maps from unitary dynamics, it is assumed that the system is initially uncorrelated from the environment.



$$\mathcal{H}_A \otimes \mathcal{H}_E = \mathcal{H}_B \otimes \mathcal{H}_{E'}$$

$$\begin{aligned}\rho_B &= \mathcal{E}_{B|A}(\rho_A) \\ &= \mathrm{Tr}_{E'}\left(U\rho_A \otimes \rho_E U^\dagger\right)\end{aligned}$$

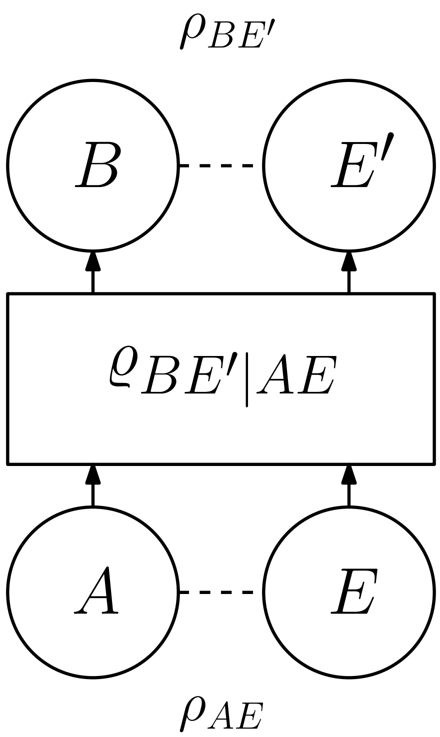
- Not a good approximation for multi-timestep dynamics with small environments, strongly correlated systems, correlated error models etc.

Does CPT dynamics hold more generally?

- Some have suggested using non CPT dynamics:
  - see e.g. T. Jordan, A. Shaji & E. Sudarshan, Phys.Rev. A 70 052110 (2004).
- More general situations in which CPT dynamics works have also been found:
  - C. Rodriguez-Rosario et. al., J. Phys. A 41 205301 (2008).
  - A. Brodutch et. al., Phys. Rev. A 87 042301 (2013).
  - F. Buscemi (2013). arXiv:1307.0363.
- From conditional states perspective, suggestion to abandon CPT is puzzling due to the analogy:

$$P(B) = \sum_A P(B|A)P(A) \qquad \rho_B = \text{Tr}_A \left(\varrho_{B|A} \rho_A \right)$$

Naive argument for CPT dynamics



$$\rho_B = \text{Tr}_{AEE'}\left(\varrho_{BE'|AE}\rho_{AE}\right)$$
$$= \text{Tr}_{A,E}\left(\varrho_{B|AE}\rho_{AE}\right)$$

$$\varrho_{ABE} = \varrho_{B|AE} \star \rho_{AE}$$

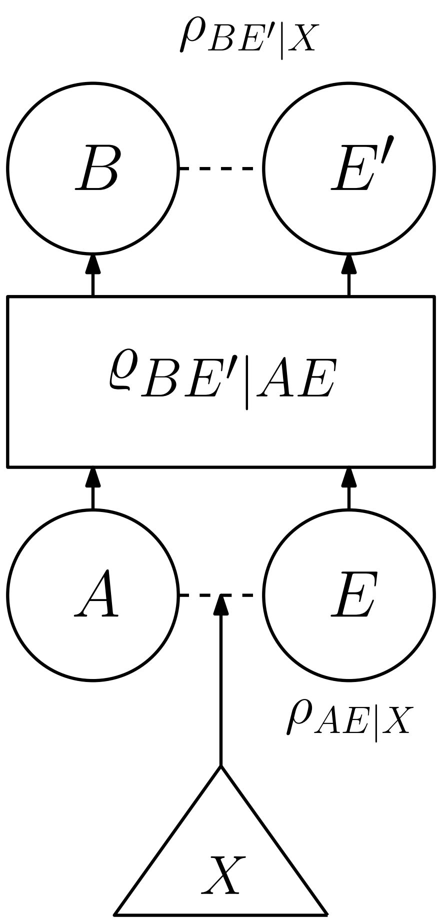
$$\varrho_{AB} = \text{Tr}_E\left(\varrho_{ABE}\right)$$

$$\rho_A = \text{Tr}_B\left(\varrho_{AB}\right)$$

$$\varrho_{B|A} = \varrho_{AB} \star \rho_A^{-1}$$

$$\rho_B = \text{Tr}_A\left(\varrho_{B|A}\rho_A\right)$$

Taking control parameters into account



$$\rho_{B|X} = \mathrm{Tr}_{AEE'}\left(\varrho_{BE'|AE}\rho_{AE|X}\right) = \mathrm{Tr}_{AE}\left(\varrho_{B|AE}\rho_{AE|X}\right)$$

$$\varrho_{ABE|X} = \varrho_{B|AE} \star \rho_{AE|X}$$

$$\varrho_{AB|X} = \mathrm{Tr}_{E}\left(\varrho_{ABE|X}\right)$$

$$\rho_{A|X} = \mathrm{Tr}_{B}\left(\varrho_{AB|X}\right)$$

$$\varrho_{B|AX} = \varrho_{AB|X} \star \rho_{A|X}^{-1}$$

$$\rho_{B|X} = \mathrm{Tr}_{A}\left(\varrho_{B|AX}\rho_{A|X}\right)$$

Conclusion

Future work

- Fundamentals:
  - Quantum conditionals problem.
  - Joint states for more general causal scenarios, e.g. multiple time-steps, pre- and post-selection.
  - General conditional independence.
- Applications:
  - Generalize other inference algorithms, e.g. MCMC. Applications in many-body physics and quantum error correction.
  - Learning algorithms.
  - Quantum master equations.

References

- Markov networks and belief propagation algorithms:
  - M. Leifer & D. Poulin, Ann. Phys. 323 1899 (2008).
- Conditional states formalism:
  - M. Leifer & R. Spekkens, Phys. Rev. A 88 052130 (2013).
  - M. Leifer & R. Spekkens, to appear in J. Phys. A (2014).
- Earlier uses of $n = 1$ conditional states:
  - M. Leifer, Phys. Rev. A 74 042310 (2006).
  - M. Leifer, AIP Conference Proceedings 889 pp. 172–186 (2007).

Additional slides

- Classically, given any valid $P(R)$, $P(S|R)$, $P(T|S,R)$ there is a joint probability distribution

$$P(R,S,T) = P(T|S,R)P(S|R)P(R)$$

that has those conditionals and marginal.

- Quantum mechanically, given any valid ρ_A, $\rho^{(n)}_{B|A}$, $\rho^{(n)}_{C|BA}$ we can certainly form

$$\rho_{ABC} = \rho_{C|AB} \star^{(n)} \left(\rho^{(n)}_{B|A} \star^{(n)} \rho_A\right),$$

but this will not necessarily have the right conditionals.

- Why? Monogamy of entanglement.

Monogamy of conditional states

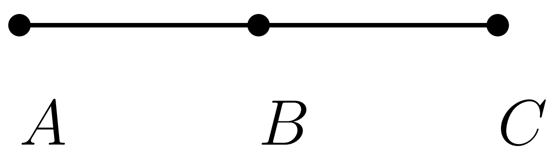
- For a pure state $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$ with

$$|\psi\rangle_{AB} = \sum_j \alpha_j \, |j\rangle_A \otimes |j\rangle_B \, ,$$

the conditional state is $\rho_{B|A} = |\psi\rangle\langle\psi|_{B|A}$, where

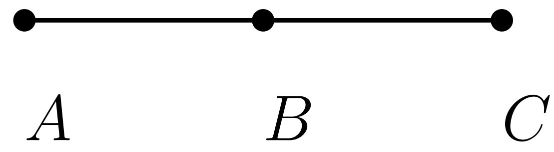
$$|\psi\rangle_{B|A} = \sum_j |j\rangle_A \otimes |j\rangle_B \, .$$

- Therefore, for a 3-system Markov Chain



cannot have both $\rho_{A|B} = |\psi\rangle\langle\psi|_{A|B}$ and $\rho_{C|B} = |\psi\rangle\langle\psi|_{C|B}$.

Monogamy for a 3-system chain



- For a 3-system Markov chain, ρ_B, $\rho_{A|B}$ and $\rho_{C|B}$ are compatilbe with a joint state iff

  1. $[\rho_{A|B}, \rho_{C|B}] = 0$. Note: this is equivalent to the existence of a decomposition of $\mathcal{H}_B$ of the form

  $$\mathcal{H}_B = \oplus_j \mathcal{H}_{B_j^A} \otimes \mathcal{H}_{B_j^C},$$

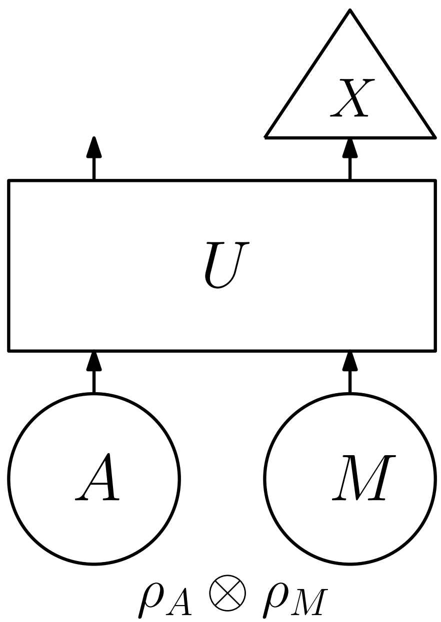
  such that

  $$\rho_{A|B} = \sum_j \rho_{A|B_j^A} \qquad \rho_{C|B_j^C} = \sum_j \rho_{C|B_j}.$$

  2. $\rho_B = \sum_j p_j \rho_{AB_j^A} \otimes \rho_{B_j^C C}$.

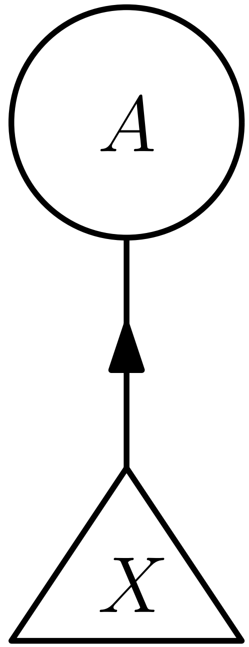
Generalized measurements

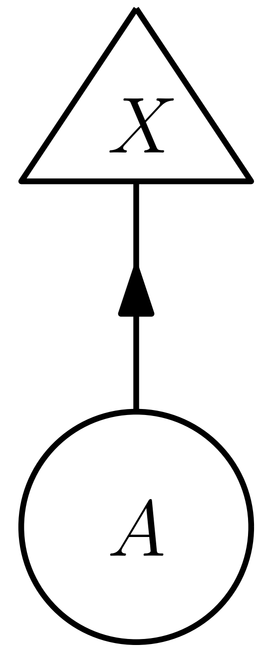
- A measurement is described by a set $\{\Pi_x\}$ of orthogonal projectors satisfying $\sum_x \Pi_x = I$.
- More generally, a measurement is described by a set $\{E_x\}$ of positive operators satisfying $\sum_x E_x = I$.
  - This is called a *Positive Operator Valued Measure (POVM)*.



$$\mathrm{Tr}_A(E_x^A \rho_A) = \mathrm{Tr}_{AM}\left(\Pi_x^M U \rho_A \otimes \rho_M U^\dagger\right)$$

Ensemble preparations and measurements





$$\rho_A = \sum_x P(X=x)\rho_x^A \qquad P(X=x) = \mathrm{Tr}_A\left(E_x^A \rho_A\right)$$

$$\rho_A = \mathrm{Tr}_X\left(\varrho_{A|X}\rho_X\right) \qquad \rho_X = \mathrm{Tr}_A\left(\varrho_{X|A}\rho_A\right)$$

ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

Hybrid conditional states

Definition. A *hybrid operator* on a classical-quantum system $\mathcal{H}_X \otimes \mathcal{H}_A$ is any operator of the form

$$M_{XA} = \sum_x |x\rangle\langle x|_X \otimes M_x^A$$

Theorem. *A hybrid (acausal or causal) conditional state of a quantum system given a classical system is any operator of the form*

$$\rho_{A|X} = \sum_x |x\rangle\langle x|_X \otimes \rho_x^A,$$

where $\{\rho_x^A\}$ is a set of density operators on $\mathcal{H}_A$.

Theorem. *A hybrid (acausal or causal) conditional state of a classical system given a quantum system is any operator of the form*

$$\rho_{X|A} = \sum_x |x\rangle\langle x|_X \otimes E_x^A,$$

where $\{E_x^A\}$ is a POVM on $\mathcal{H}_A$.

ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

Summary so far

	Conventional notation	**Conditional states**
Probability distribution of X	$P(X)$	ρ_X
Set of states on A	$\{\rho_x^A\}$	$\rho_{A\|X}$
POVM on A	$\{E_x^A\}$	$\rho_{X\|A}$
CPT map from A to B	$\mathcal{E}_{B\|A}$	$\varrho_{B\|A}$
Ensemble averaging	$\rho_A = \sum_x P(X=x)\rho_x^A$	$\rho_A = \mathrm{Tr}_X\left(\rho_{A\|X}\rho_X\right)$
Born rule	$P(X=x) = \mathrm{Tr}_A\left(E_x^A \rho_A\right)$	$\rho_X = \mathrm{Tr}_A\left(\rho_{X\|A}\rho_A\right)$
Action of a CPT map	$\rho_B = \mathcal{E}_{B\|A}(\rho_A)$	$\rho_B = \mathrm{Tr}_A\left(\varrho_{B\|A}\rho_A\right)$

Quantum Bayes' theorem

- For acausal states we have

$$\begin{aligned}\rho_{AB} &= \rho_{B|A} \star \rho_A \\ &= \rho_{A|B} \star \rho_B,\end{aligned}$$

and hence

$$\rho_{A|B} = \rho_{B|A} \star \left(\rho_A \otimes \rho_B^{-1}\right)$$

- In the causal case, we can define

$$\varrho_{A|B} = \varrho_{B|A} \star \left(\rho_A \otimes \rho_B^{-1}\right),$$

so that

$$\begin{aligned}\varrho_{AB} &= \varrho_{B|A} \star \rho_A \\ &= \varrho_{A|B} \star \rho_B.\end{aligned}$$

ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

- The "pretty good" measurement:

$$\rho_{X|A} = \rho_{A|X} \star \left(\rho_X \otimes \rho_A^{-1}\right) \qquad E_x^A = P(X = x)\rho_A^{-\frac{1}{2}} \rho_x^A \rho_A^{-\frac{1}{2}}$$

- Remote state collapse:

$$\rho_{A|X} = \rho_{X|A} \star \left(\rho_A \otimes \rho_X^{-1}\right) \qquad \rho_x^A = \frac{\rho_A^{\frac{1}{2}} E_x^A \rho_A^{\frac{1}{2}}}{\mathrm{Tr}_A\left(E_x^A \rho_A\right)}$$

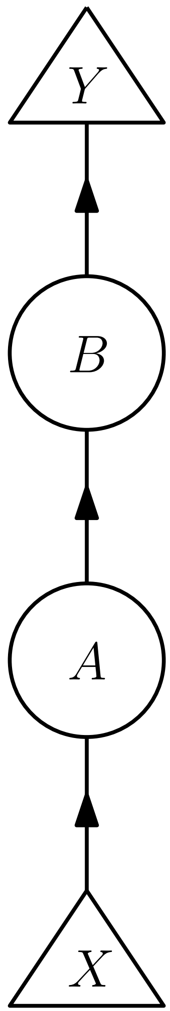
- Barnum-Knill approximate error correction:

$$\varrho_{A|B} = \varrho_{B|A} \star \left(\rho_A \otimes \rho_B^{-1}\right)$$

$$\mathcal{E}_{A|B}(\cdot) = \rho_A^{\frac{1}{2}} \otimes \rho_B^{-\frac{1}{2}} \mathcal{E}_{B|A}(\cdot) \rho_A^{\frac{1}{2}} \otimes \rho_B^{-\frac{1}{2}}$$

P. Hausladen & W. Wootters, J. Mod. Opt. 41 2385 (1994).
C. Fuchs, J. Mod. Opt. 50 987 (2003).
H. Barnum & E. Knill, J. Math. Phys. 43 2097 (2002).
ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

Retrodictive quantum theory



- Predictive expression:

$$P(X,Y) = \mathrm{Tr}_{AB}\left(\rho_{Y|B}\varrho_{B|A}\rho_{A|X}\rho_X\right)$$

- Retrodictive expression:

$$P(X,Y) = \mathrm{Tr}_{AB}\left(\rho_{X|A}\varrho_{A|B}\rho_{B|Y}\rho_Y\right)$$

- Converted into conventional notation, this generalizes S. Barnett, D. Pegg & J. Jeffers, J. Mod. Opt. 47 1779 (2000).

ML & R. Spekkens, Phys. Rev. A 88 052130 (2013).

Nonlinear dynamics

- Classically, $P(E|A,X) = P(E|A)$ is sufficient for $P(B|A,X) = P(A|X)$.
- In quantum theory, $\rho_{E|AX} = \rho_{E|A}$ is not sufficient for $\varrho_{B|AX} = \varrho_{B|A}$ due to nonlinearity and noncommutativity of conditional states.
- Instead one obtains

$$\rho_{B|X} = \mathrm{Tr}_{AE}\left(\varrho_{B|AE}\rho_{A|X}^{\frac{1}{2}}\rho_{E|A}\rho_{A|X}^{\frac{1}{2}}\right),$$

  which is nonlinear in $\rho_{A|X}$.
- Suggests nonlinear maps may have physically relevant applications.