Quantum Dynamics as Generalized Conditional Probabilities

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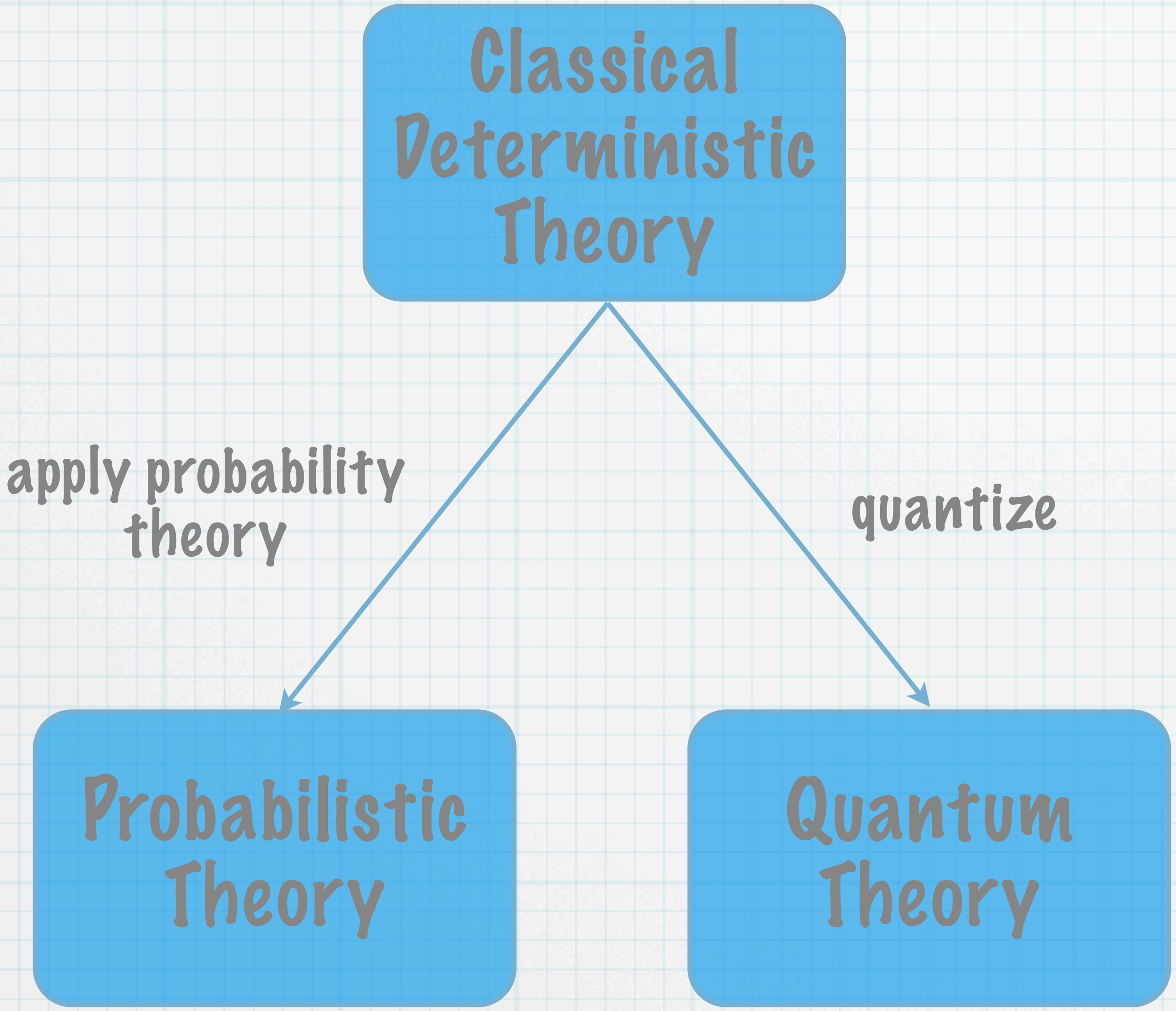
Quantum Theory as a Meta-theory



* The analogy is based on quantum measurement theory....
* ...but QT is not as abstract as PT. Quantization cannot be applied to an arbitrary theory.
* Causal structure is present in QT, but absent from PT.

Classical Probability vs. Quantum Theory

Classical		Quantum	
Probability distribution:	$P(X)$	Quantum State:	ρ_A
Joint distribution:	$P(X,Y)$	Joint State:	ρ_{AB}
Transition matrix:	$\Gamma_{Y\|X}$	TPCP map:	$\mathcal{E}_{B\|A}$
Conditional Prob.:	$P(Y\|X)$	?	

Why quantum conditional probability?

* Conditional probabilities allow all types of correlation to be treated on an equal footing, whether timelike, spacelike or completely abstract.
    * Causal relations are not primitive in probability theory.
* Some classical probabilistic structures are defined in terms of conditional probability.
    * Markov Chains
    * Bayesian Networks
* Some Bayesians take conditional probability to be the most fundamental notion.
    * See textbook by D. V. Lindley

Outline

1. Introduction
   i. The Many faces of conditional probability
   ii. Suggestions for a quantum analog of conditional probability
2. Stochastic Dynamics as Conditional Probabilities
3. Choi-Jamiolkowski Isomorphism
4. A New Isomorphism
5. Operational Interpretation
6. Application: Cloning, broadcasting & monogamy of entanglement
7. Future Directions

1. Introduction

(A) Reconstructing a joint distribution from a marginal

$$P(X, Y) = P(Y|X)P(X)$$

(B) Bayesian Updating

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

(C) Stochastic Dynamics

$$P(Y = i) = \sum_j (\Gamma_{Y|X})_{ij} P(X = j)$$

(D) Conditional Shannon Entropy

$$H(Y|X) = -\sum_{X,Y} P(X, Y) \log_2 P(Y|X)$$

(E) Reduction of complexity via conditional independence

$$P(Y|X, Z) = P(Y|Z) \Leftrightarrow P(X, Y, Z) = P(X|Z)P(Y|Z)P(Z)$$

1. Introduction

(A) Reconstruction of a joint state ρ_{AB} from a marginal ρ_A.

(B) Updating quantum states after a measurement

$$\rho_{|M} = \frac{\mathcal{E}^M(\rho)}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$

(C) TPCP dynamics

$$\rho_B = \mathcal{E}_{B|A}\left(\rho_A\right)$$

(D) Conditional von Neumann Entropy

$$S(B|A) = -\mathrm{Tr}\left(\rho_{AB}\log_2 \rho_{B|A}\right)$$

(E) Reduction of complexity via conditional independence

1. Introduction

- Cerf & Adami ('97-'99):

$$\begin{aligned}\rho_{B|A} &= 2^{\log_2 \rho_{AB} - \log_2 \rho_A \otimes I_B} \\ &= \lim_{n\to\infty}\left[\rho_{AB}^{\frac{1}{n}}\left(\rho_A \otimes I_B\right)^{\frac{1}{n}}\right]^n\end{aligned}$$

- (A) Reconstruction:

$$\rho_{AB} = 2^{\log_2 \rho_A \otimes I_B + \log_2 \rho_{B|A}}$$

- (C) Entropy:

$$S(B|A) = S(A,B) - S(A) = -\mathrm{Tr}\left(\rho_{AB}\log_2 \rho_{B|A}\right)$$

- (E) Complexity Reduction: If $\log_2 \rho_{B|AC} = I_A \otimes \log_2 \rho_{B|C}$

$$\rho_{ABC} = 2^{I_{AB}\otimes \log_2 \rho_C + \log_2 \rho_{A|C}\otimes I_B + I_A \otimes \log_2 \rho_{B|C}}$$

1. Introduction

* (B) Updating:
  * POVM: $$M = \{\boldsymbol{M}\}, \quad \boldsymbol{M} > 0, \quad \sum_M \boldsymbol{M} = I$$
  * Probability Rule: $$P(M) = \mathrm{Tr}\left(\boldsymbol{M}\rho\right)$$
  * Update CP-map: $$\rho_{|M} = \frac{\mathcal{E}^M(\rho)}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$

$$\mathcal{E}^M(\rho) = \sum_j A_j^M \rho A_j^{M\dagger} \qquad \sum_j A_j^{M\dagger} A_j^M = \boldsymbol{M}$$

  * $\mathcal{E}^M$ depends on details of system-measuring device interaction.

1. Introduction

* Is there one update rule that is more "Bayes' rule like" than the rest?
    * Traditionally (see Bub '77 for projective measurements):

$$\rho_{|M} = \frac{\sqrt{\boldsymbol{M}}\rho\sqrt{\boldsymbol{M}}}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$

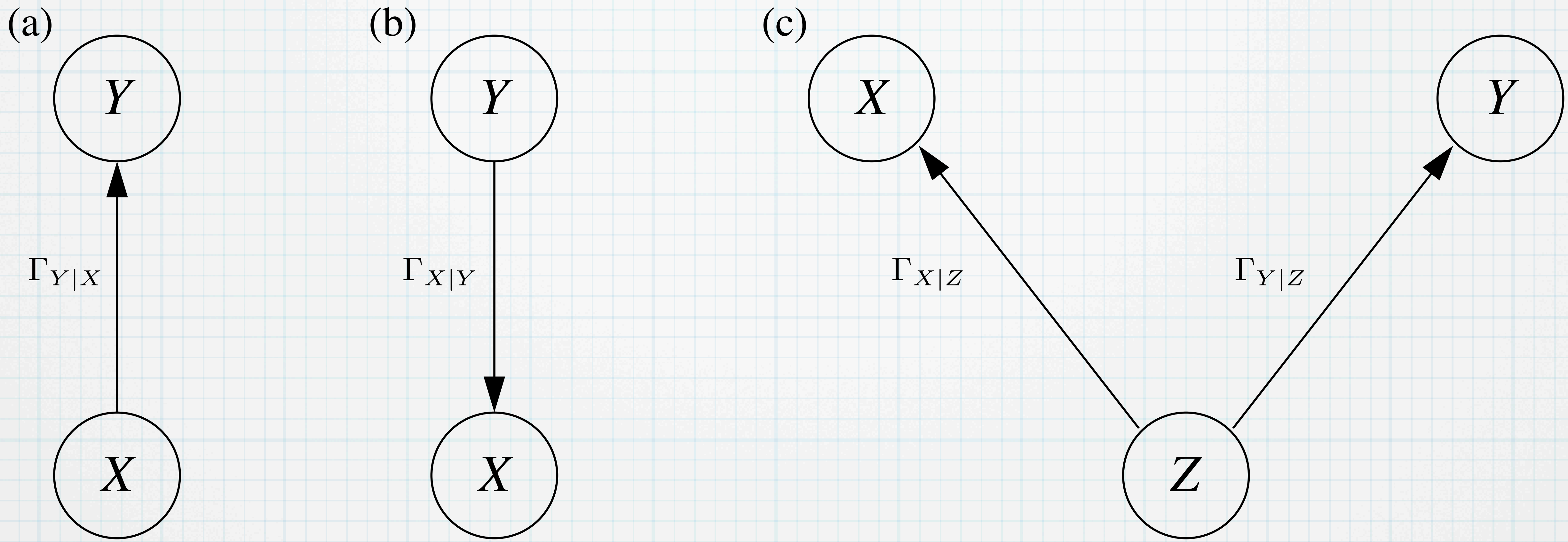
    * According to Fuchs ('01, '02):

$$\rho_{|M} = \frac{\sqrt{\rho}\boldsymbol{M}\sqrt{\rho}}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$

    * Both reduce to Bayes' rule when the $\boldsymbol{M}$ are projection operators and

$$[\boldsymbol{M}, \rho] = 0$$

2. Dynamics as conditional probability



$$P(Y) = \sum_X \Gamma_{Y|X} P(X)$$

$$P(X,Y) = \Gamma_{Y|X} P(X)$$

$$P(Y|X) = \Gamma^r_{Y|X}$$

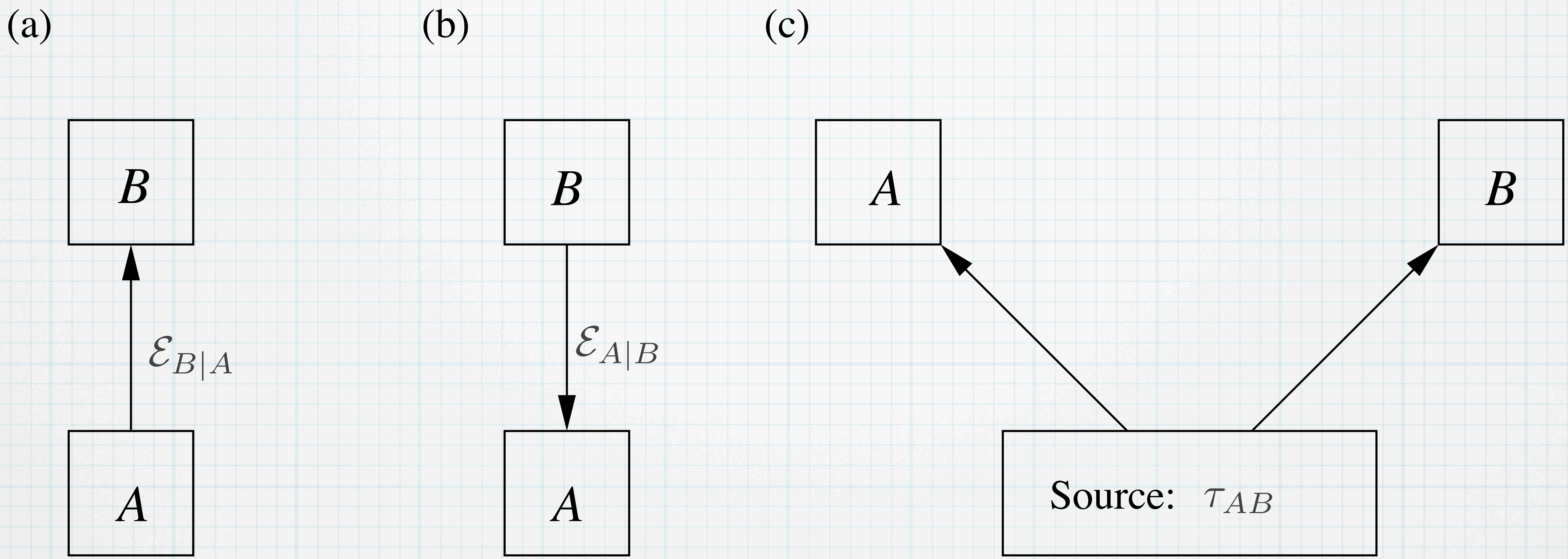
$$P(X,Y)$$

$$P(X) = \sum_Y P(X,Y)$$

$$P(Y|X) = \frac{P(X,Y)}{P(X)}$$

Isomorphism: $(P(X), \Gamma^r_{Y|X}) \quad \Leftrightarrow \quad P(X,Y)$

2. Dynamics as conditional probability



$$\rho_B = \mathcal{E}_{B|A}\left(\rho_A\right)$$

$$\tau_{AB}$$

$$\rho_{AB} = ?$$

$$\tau_A = \mathrm{Tr}_B\left(\tau_{AB}\right)$$

$$\rho_{B|A}, \mathcal{E}^r_{B|A}, ?$$

$$\tau_{B|A} = ?$$

Isomorphism: $\left(\rho_A, \mathcal{E}^r_{B|A}\right) \quad \Leftrightarrow \quad \tau_{AB}$ **?**

3. Choi-Jamiolkowski Isomorphism

* For bipartite pure states and operators:

$$R_{B|A} = \sum_{jk} \alpha_{jk} \left|j\right\rangle_B \left\langle k\right|_A \Leftrightarrow \left|\Psi\right\rangle_{AB} = \sum_{jk} \alpha_{jk} \left|k\right\rangle_A \otimes \left|j\right\rangle_B$$

* For mixed states and CP-maps:

$$\mathcal{E}_{B|A}\left(\rho_A\right) = \sum_{\mu} R^{(\mu)}_{B|A} \rho_A R^{(\mu)\dagger}_{B|A} \Rightarrow \tau_{AB} = \sum_{\mu} \left|\Psi^{(\mu)}\right\rangle_{AB} \left\langle \Psi^{(\mu)}\right|_{AB}$$

3. Choi-Jamiolkowski Isomorphism

* Let $\left|\Phi^+\right\rangle_{AA'} = \frac{1}{\sqrt{d_A}} \sum_j |j\rangle_A \otimes |j\rangle_{A'}$

* Then $\tau_{AB} = \mathcal{I}_A \otimes \mathcal{E}_{B|A'}\left(\left|\Phi^+\right\rangle_{AA'}\left\langle\Phi^+\right|_{AA'}\right)$

$$\mathcal{E}_{B|A}\left(\rho_A\right) = d_A^2 \left\langle\Phi^+\right|_{AA'} \rho_A \otimes \tau_{A'B} \left|\Phi^+\right\rangle_{AA'}$$

* Operational interpretation: Noisy gate teleportation.

3. Choi-Jamiolkowski Isomorphism

* Remarks:
    * Isomorphism is basis dependent. A basis must be chosen to define $\left|\Phi^+\right\rangle_{AA'}$.
    * If we restrict attention to Trace Preserving CP-maps then

$$\tau_A = \mathrm{Tr}_B\left(\tau_{AB}\right) = \frac{1}{d_A} I_A$$

    * This is a special case of the isomorphism we want to construct

$$\left(\rho_A, \mathcal{E}^r_{B|A}\right) \quad \Leftrightarrow \quad \tau_{AB}$$

where $\rho_A = \dfrac{1}{d_A} I_A$.

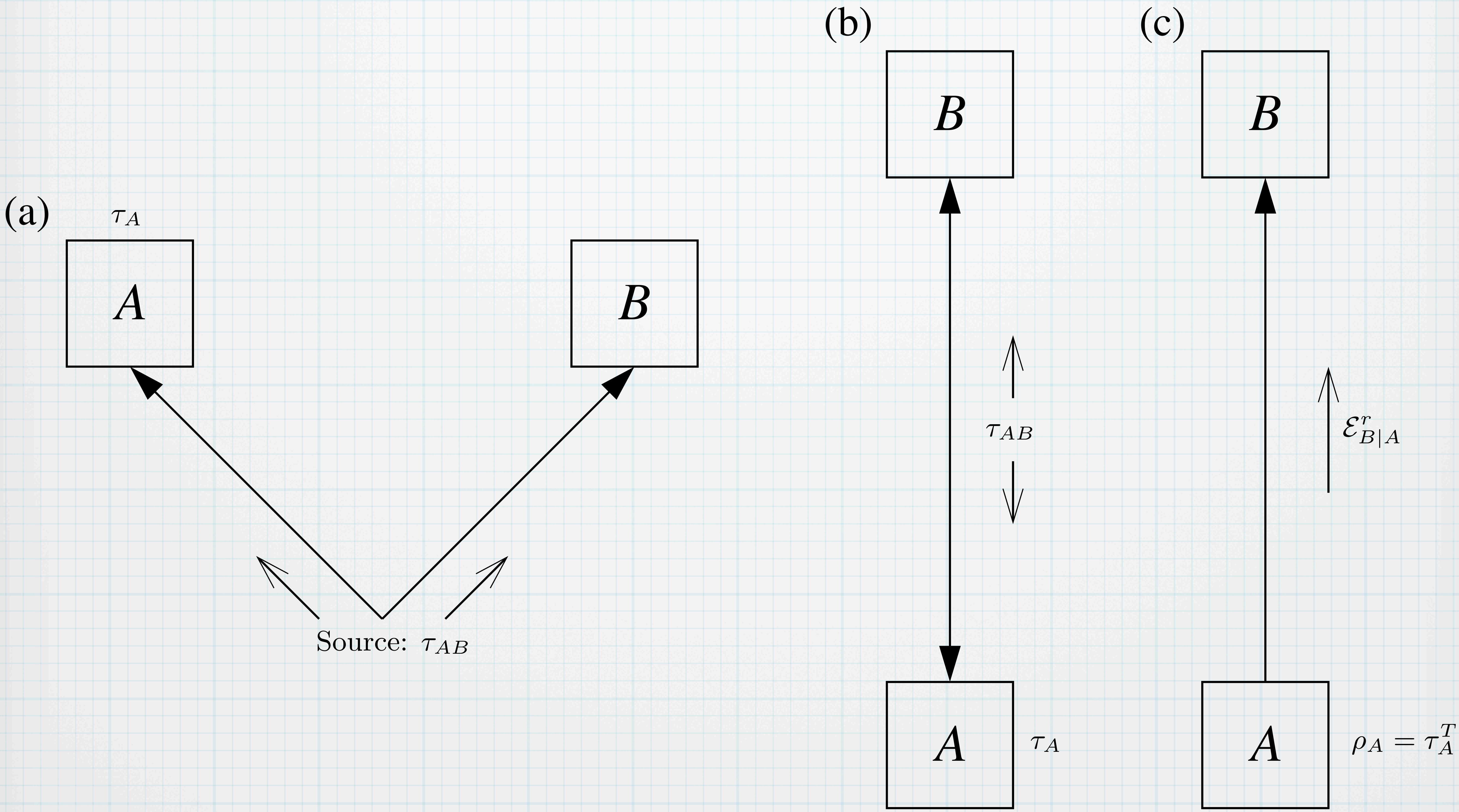
4. A New Isomorphism

* $\left(\rho_A, \mathcal{E}^r_{B|A}\right) \rightarrow \tau_{AB}$ direction:
  * Instead of $\left|\Phi^+\right\rangle_{AA'}$ use $\left|\Phi\right\rangle_{AA'} = \left(\rho_A^T\right)^{\frac{1}{2}} \otimes I_{A'} \left|\Phi^+\right\rangle_{AA'}$
  * Then $\tau_{AB} = \mathcal{I}_A \otimes \mathcal{E}^r_{B|A'} \left(\left|\Phi\right\rangle_{AA'} \left\langle\Phi\right|_{AA'}\right)$

* $\tau_{AB} \rightarrow \left(\rho_A, \mathcal{E}^r_{B|A}\right)$ direction:
  * Set $\rho_A = \tau_A^T, \qquad \tau_A = \mathrm{Tr}_B\left(\tau_{AB}\right)$
  * Let $\sigma_{B|A} = \tau_A^{-\frac{1}{2}} \otimes I_B \tau_{AB} \tau_A^{-\frac{1}{2}} \otimes I_B$
  * $\sigma_{B|A}$ is a density operator, satisfying $\mathrm{Tr}_B\left(\sigma_{B|A}\right) = \dfrac{1}{d^r_A} P_A$
  * It is uniquely associated to a TPCP map $\mathcal{E}^r_{B|A} : \mathfrak{L}\left(P_A \mathcal{H}_A\right) \rightarrow \mathfrak{L}\left(\mathcal{H}_B\right)$

    via the Choi-Jamiolkowski isomorphism.

4. A New Isomorphism



5. Operational Interpretation

- Reminder about measurements:
  - POVM: $M = \{\boldsymbol{M}\}, \quad \boldsymbol{M} > 0, \quad \sum_{M} \boldsymbol{M} = I$
  - Probability Rule: $P(M) = \mathrm{Tr}\,(\boldsymbol{M}\rho)$
  - Update CP-map: $\rho_{|M} = \dfrac{\mathcal{E}^M(\rho)}{\mathrm{Tr}\,(\boldsymbol{M}\rho)}$

$$\mathcal{E}^M(\rho) = \sum_j A_j^M \rho A_j^{M\dagger} \qquad \sum_j A_j^{M\dagger} A_j^M = \boldsymbol{M}$$

  - $\mathcal{E}^M$ depends on details of system-measuring device interaction.

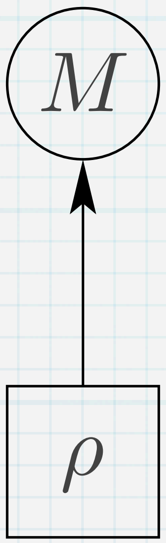
5. Operational Interpretation

* **Lemma:** $\rho = \sum_{M} P(M) \rho_{|M}$ is an ensemble decomposition of a density matrix ρ iff there is a POVM $M = \{\boldsymbol{M}\}$ s.t.

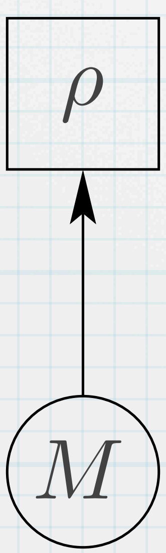
$$P(M) = \mathrm{Tr}\left(\boldsymbol{M}\rho\right) \qquad \rho_{|M} = \frac{\sqrt{\rho}\boldsymbol{M}\sqrt{\rho}}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$

* **Proof sketch:** $\boldsymbol{M} = P(M)\rho^{-\frac{1}{2}}\rho_{|M}\rho^{-\frac{1}{2}}$

5. Operational Interpretation

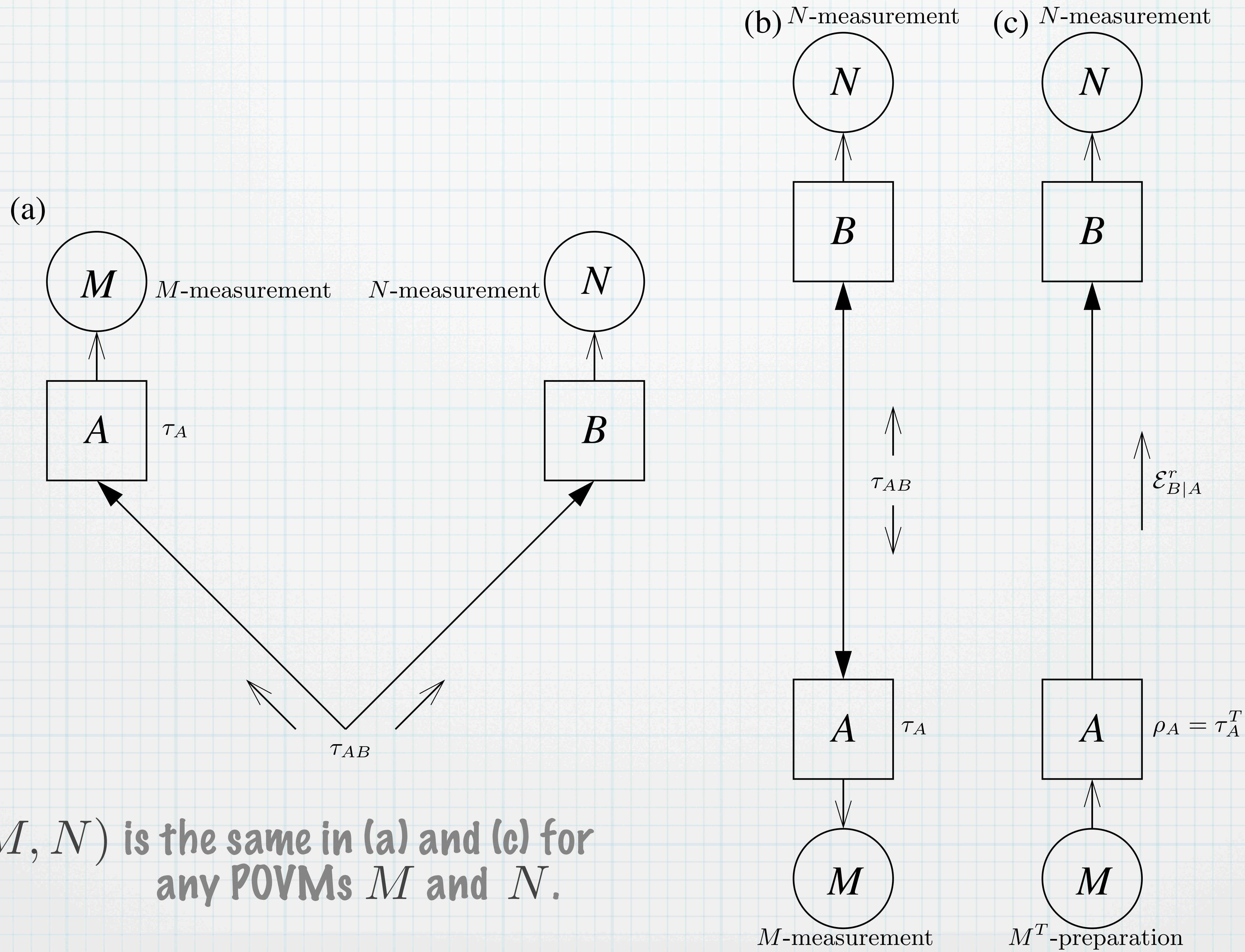


* M-measurement of ρ
  * Input: ρ
  * Measurement probabilities: $P(M) = \mathrm{Tr}\left(\boldsymbol{M}\rho\right)$
  * Updated state:
$$\rho_{|M} = \frac{\sqrt{\boldsymbol{M}}\rho\sqrt{\boldsymbol{M}}}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$



* M-preparation of ρ
  * Input: Generate a classical r.v. with p.d.f
$$P(M) = \mathrm{Tr}\left(\boldsymbol{M}\rho\right)$$
  * Prepare the corresponding state:
$$\rho_{|M} = \frac{\sqrt{\rho}\boldsymbol{M}\sqrt{\rho}}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$

5. Operational Interpretation



$P(M, N)$ is the same in (a) and (c) for any POVMs M and N.

6. Application: Broadcasting & Monogamy

* For any TPCP map $\mathcal{E}_{BC|A} : \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_B \otimes \mathcal{H}_C)$ the reduced maps are:

$$\mathcal{E}_{B|A} = \mathrm{Tr}_C \circ \mathcal{E}_{BC|A} \qquad \mathcal{E}_{C|A} = \mathrm{Tr}_B \circ \mathcal{E}_{BC|A}$$

* The following commutativity properties hold:

$$\begin{tikzcd}
\rho_{ABC} \arrow[r, equal] \arrow[d, "\mathrm{Tr}_C"'] & (\rho_A, \mathcal{E}^r_{BC|A}) \arrow[d, "\mathrm{Tr}_C"] \\
\rho_{AB} \arrow[r, equal] & (\rho_A, \mathcal{E}^r_{B|A}).
\end{tikzcd}$$

* Therefore, 2 states ρ_{AB}, ρ_{AC} incompatible with being the reduced states of a global state ρ_{ABC}.

* 2 reduced maps $\mathcal{E}^r_{B|A}, \mathcal{E}^r_{C|A}$ incompatible with being the reduced maps of a global map $\mathcal{E}^r_{BC|A}$.

6. Application: Broadcasting & Monogamy

* A TPCP-map $\mathcal{E}_{A'A''|A} : \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_{A'} \otimes \mathcal{H}_{A''})$ is broadcasting for a state ρ_A if

$$\mathcal{E}_{A'|A}(\rho_A) = \rho_{A'} \qquad \mathcal{E}_{A''|A}(\rho_A) = \rho_{A''}$$

* A TPCP-map $\mathcal{E}_{A'A''|A} : \mathfrak{L}(\mathcal{H}_A) \to \mathfrak{L}(\mathcal{H}_{A'} \otimes \mathcal{H}_{A''})$ is cloning for a state ρ_A if

$$\mathcal{E}_{A'A''|A}(\rho_A) = \rho_{A'} \otimes \rho_{A''}$$

* Note: For pure states cloning = broadcasting.
* A TPCP-map is universal broadcasting if it is broadcasting for every state.

6. Application: Broadcasting & Monogamy

* No cloning theorem (Dieks '82, Wootters & Zurek '82):
    * There is no map that is cloning for two nonorthogonal and nonidentical pure states.
* No broadcasting theorem (Barnum et. al. '96):
    * There is no map that is broadcasting for two noncommuting density operators.
* Clearly, this implies no universal broadcasting as well.
* Note that the maps $\mathcal{E}_{A'|A}, \mathcal{E}_{A''|A}$ are valid individually, but they cannot be the reduced maps of a global map $\mathcal{E}_{A'A''|A}$.

6. Application: Broadcasting & Monogamy

* The maps $\mathcal{E}_{A'|A}, \mathcal{E}_{A''|A}$ must be related to incompatible states $\tau_{AA'}, \tau_{AA''}$

* Theorem: If $\mathcal{E}_{A'A''|A}$ is universal broadcasting, then both $\tau_{AA'}, \tau_{AA''}$ must be pure and maximally entangled.

* Ensemble broadcasting $\{(p, \rho_1), ((1-p), \rho_2)\}$ s.t. $[\rho_1, \rho_2] \neq 0$

$$\left(p\rho_1 + (1-p)\rho_2, \mathcal{E}^r_{A'A''|A}\right) \Leftrightarrow \tau_{AA'A''}$$

* Theorem: There is a local operation on A that transforms both $\tau_{AA'}$ and $\tau_{AA''}$ into pure, entangled states with nonzero probability of success.

7. Future Directions

* Quantitative relations between approximate ensemble broadcasting and monogamy inequalities for entanglement.

* More generally, useful in analyzing any qinfo protocol involving the action of a TPCP-map on a particular ensemble rather than the whole Hilbert space.

* Can the various analogs of conditional probability be unified?

* Can quantum theory be developed using an analog of conditional probability as the fundamental notion?

* Can we eliminate background causal structures entirely from the formalism of quantum theory?