Unentangled Bit Commitment and the Clifton-Bub-Halvorson (CBH) Theorem

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Outline

The Brassard-Fuchs Speculation

In ≈2000, Brassard and Fuchs speculated that the basic Hilbert Space structures of quantum theory might be uniquely determined by two cryptographic constraints:

- The Possibility of Secure Key Distribution
- The Impossibility of Bit Commitment

This was to be viewed as analogous to Einstein's derivation of the kinematics for special relativity from the two postulates:

- The laws of physics are invariant under changes of inertial frame.
- The speed of light in vacuo is constant in all inertial frames.

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This derivation has to be done within a precise mathematical framework for physical theories, which must be:

- Narrow enough to convert the axioms into precise mathematical constraints.
- Broad enough that the work is being done by the postulates rather than the framework assumptions.

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The CBH Theorem

In 2003, Clifton, Bub and Halvorson "derived quantum theory" from:

- The impossibility of superluminal information transfer between two physical systems by performing measurements on one of them.
- The impossibility of perfectly broadcasting the information contained in an unknown physical state.
- The impossibility of unconditionally secure bit commitment.

The mathematical framework chosen was C^*-algebraic theories.

The CBH Theorem

CBH don't arrive exactly at quantum theory, but intend their theorem to be read as follows:

1. No signalling $\Rightarrow$ Separate systems correspond to commuting algebras of observables.
2. No broadcasting $\Rightarrow$ Algebras corresponding to individual systems are nonabelian.
3. No bit commitment $\Rightarrow$ Bipartite systems can occupy entangled states.

There is some debate about whether 3 is independent of 1 and 2.

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Why C^*-algebras?

We are not in the business of rigorous axiomatization, so CBH say:

> *...it suffices for present purposes simply to observe that all physical theories that have been found empirically successful – not just phase space and Hilbert space theories but also theories based on a manifold – fall under this framework*

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C^*-algebras: Reasons to be skeptical

- C^*-algebras were invented to do a Hilbert's 10th job on quantum theory – particularly QFT and quantum stat. mech.
- Every C^*-algebra has a faithful Hilbert space representation (GNS theorem).
- In finite dimensions we only have classical probability, quantum theory and quantum theory with superselection rules.
- In infinite dimensions it's essentially the same story.

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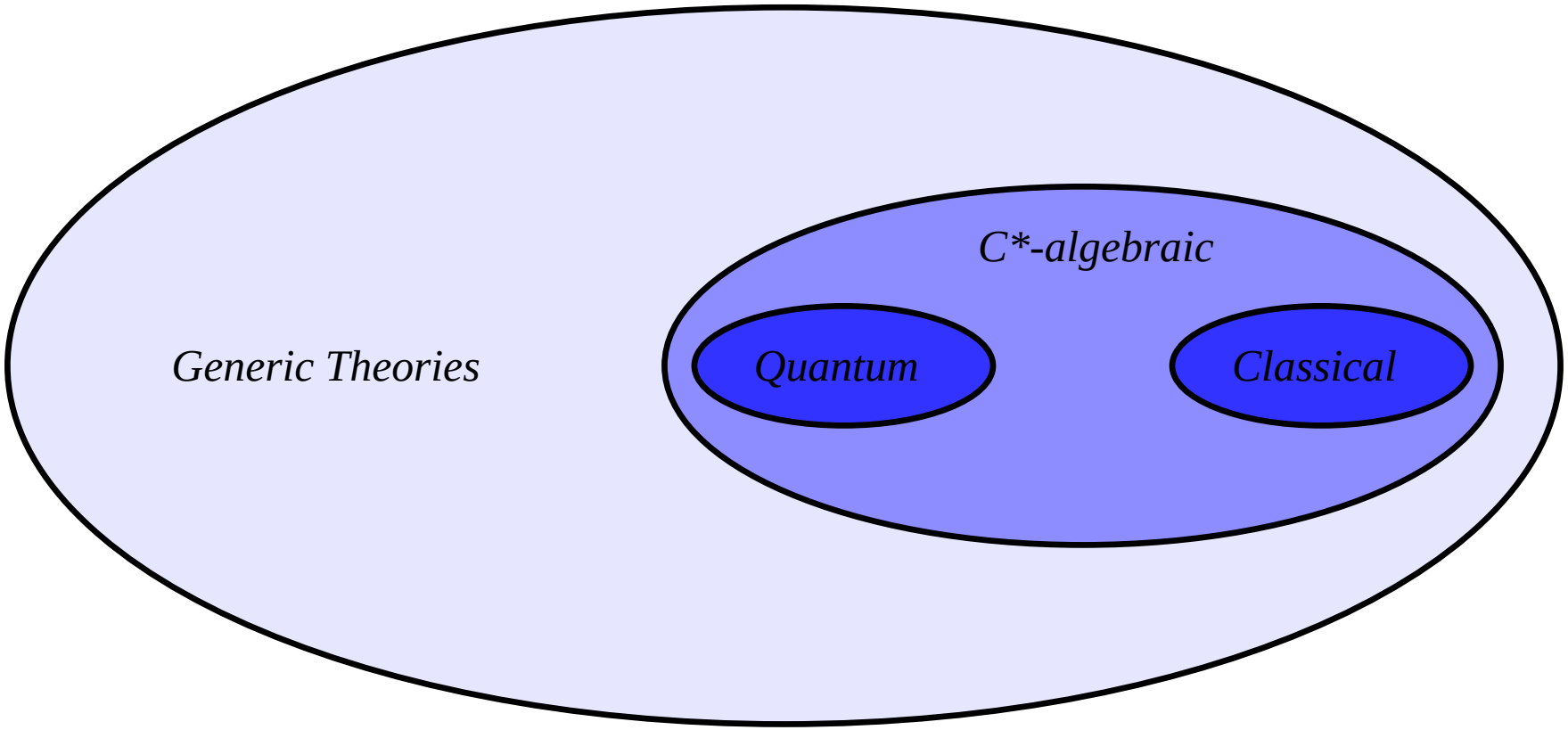
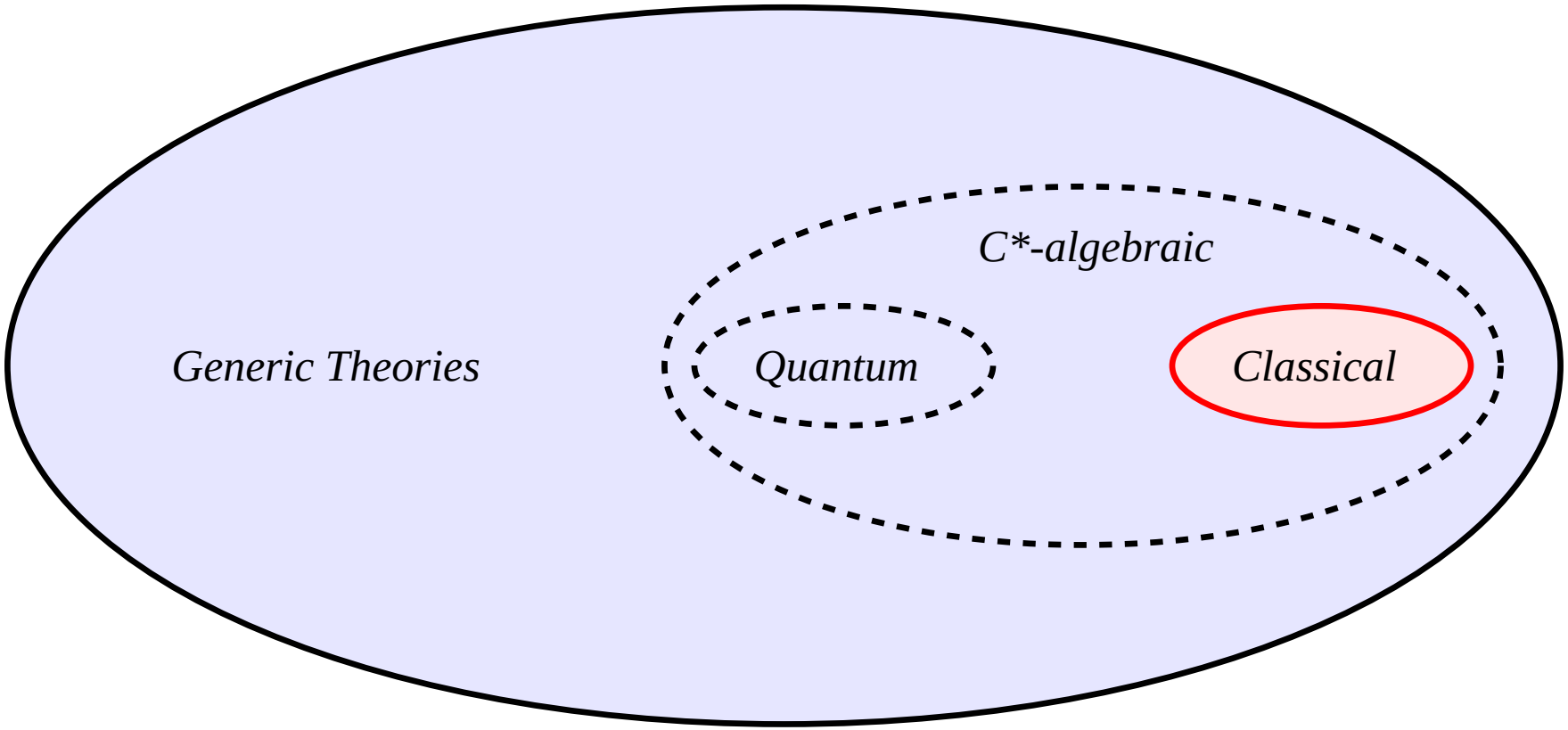
Generalized Probabilistic Frameworks

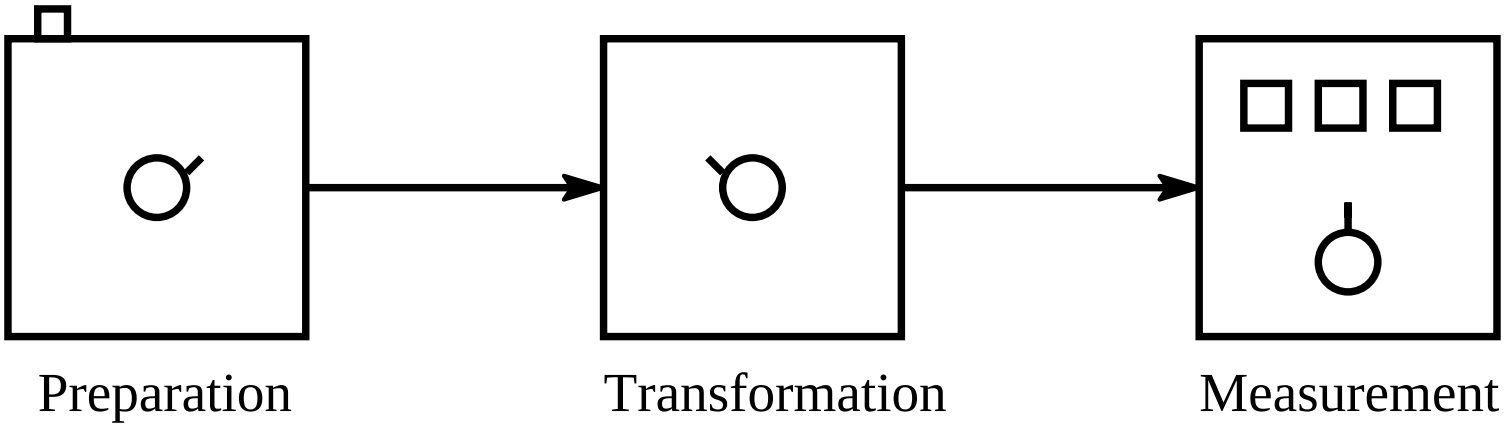


The End Result



The Convex Sets Framework

- A traditional operational framework.



- Goal: Predict Prob(outcome|Choice of P, T and M)

Preparations $\longrightarrow$ States

Definition

The set Ω of normalized states is a compact, closed, convex set.

- Convex: If $\omega, \mu \in \Omega$ and $p \in [0,1]$ then $p\omega + (1-p)\mu \in \Omega$.
- Extreme points of Ω are called pure states.
- Note: Every convex subset of a locally convex topological vector space is affinely homeomorphic to the set of all states on a test space (F. W. Shultz, *Journal of Combinatorial Theory A* 17, 317 (1974)).

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Examples

- Classical: $\Omega =$ Probability simplex.
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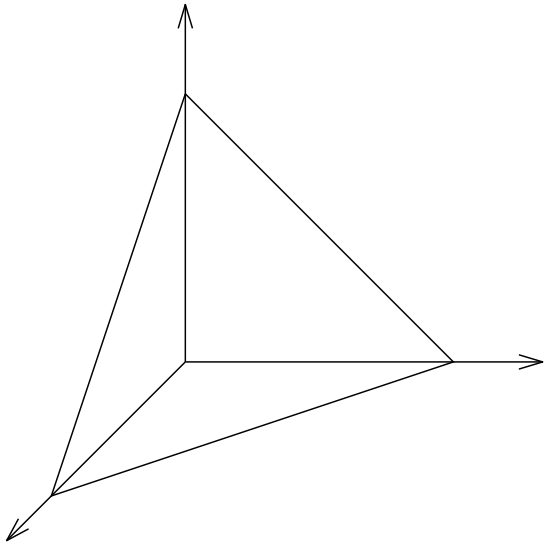
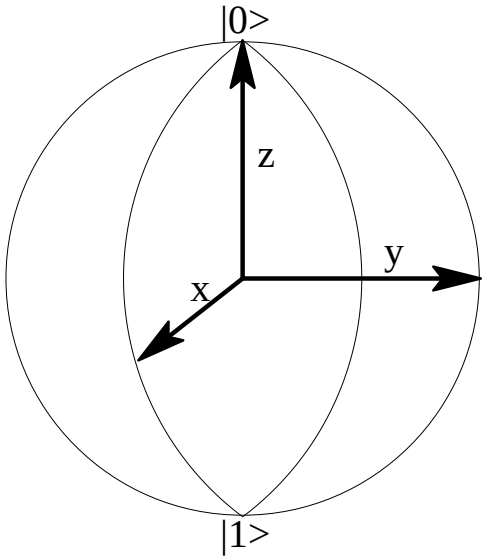
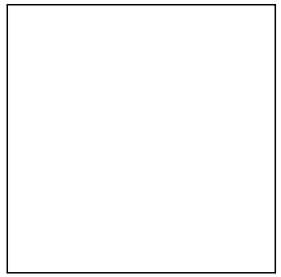
- Polyhedral.







Measurement Outcomes → Effects

Definition

Let $A(\Omega)$ be the set of affine functionals $\Omega \to \mathbb{R}$ and $V(\Omega)$ be the set of positive affine functionals $\Omega \to \mathbb{R}^+$.

$$\forall p \in [0,1],\ f(p\omega + (1-p)\mu) = pf(\omega) + (1-p)f(\mu)$$

- $A(\Omega)$ is a vector space and $V(\Omega)$ is a convex cone.

$$(\alpha f + \beta g)(\omega) = \alpha f(\omega) + \beta g(\omega)$$

- $V(\Omega)$ spans $A(\Omega)$.
- Partial order on $A(\Omega)$: $f \leq g$ iff $\forall \omega \in \Omega,\ f(\omega) \leq g(\omega)$.
- Unit: $\forall \omega \in \Omega,\ \tilde{1}(\omega) = 1$. Zero: $\forall v \in V,\ \tilde{0}(v) = 0$.
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States as vectors

- Consider the dual space $A(\Omega)^*$ of linear functionals $A(\Omega) \longrightarrow \mathbb{R}$ and the dual cone of $V(\Omega)^*$ of linear functionals $V(\Omega) \longrightarrow \mathbb{R}^+$.
- $V(\Omega)^*$ can be extended to $A(\Omega)$.
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Examples

- Classical:
  - $A(\Omega) = \{\text{functions}\}$
  - $V(\Omega) = \{\text{positive functions}\}$
  - $[\tilde{0}, \tilde{1}] = \{\text{Fuzzy indicator functions}\}$
  - $V(\Omega)^* = \{\text{positive functions}\}$
- Quantum:
  - $A(\Omega) \cong \{\text{Hermitian operators}\}$ via $f(\rho) = \text{Tr}(A_f \rho)$
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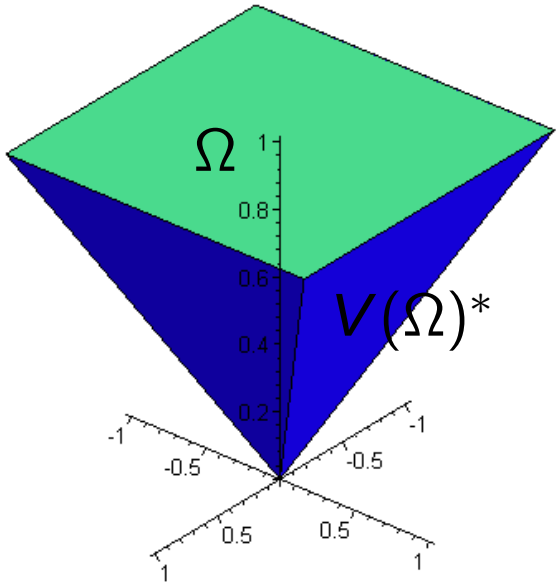

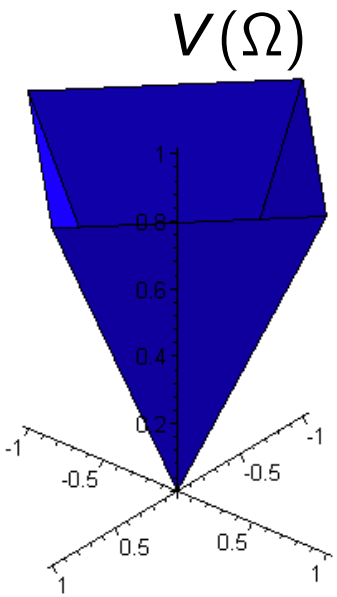

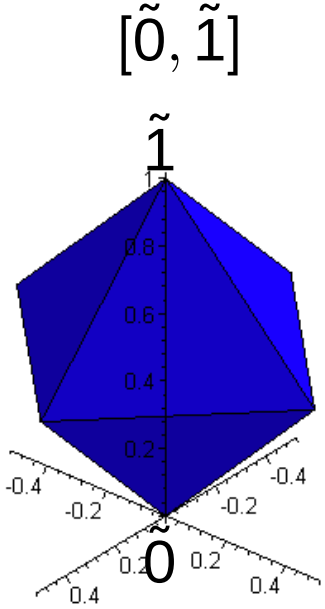

Observables

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An observable is a finite collection $(f_1, f_2, \ldots, f_N)$ of elements of $[\tilde{0}, \tilde{1}]$ that satisfies $\sum_{j=1}^{N} f_j = \tilde{1}$.

- Note: Analogous to a POVM in Quantum Theory.
- Can give more sophisticated measure-theoretic definition.

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Dynamics

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The dynamical maps $\mathfrak{D}_{B|A}$ are a convex subset of the affine maps $\phi : V(\Omega_A)^* \to V(\Omega_B)^*$.

$$\forall \alpha, \beta \geq 0,\ \phi(\alpha\omega_A + \mu_B) = \alpha\phi(\omega_A) + \beta\phi(\mu_B)$$

You might want to require other things, e.g.

- The identity is in $\mathfrak{D}_{A|A}$.
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Combining Systems: Tensor Products

- Given Ω_A and Ω_B, what is the joint space Ω_{AB}?
- We assume:
  - A joint state must assign joint probabilities to $f_A \in [\tilde{0}_A, \tilde{1}_B], f_B \in [\tilde{0}_A, \tilde{1}_B]$.
  - No-signaling.
  - States are uniquely determined by probability assignments to pairs f_A, f_B.
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Combining Systems: Tensor Products

- Given Ω_A and Ω_B, what is the joint space Ω_{AB}?
- We assume:
  - A joint state must assign joint probabilities to $f_A \in [\tilde{0}_A, \tilde{1}_B], f_B \in [\tilde{0}_A, \tilde{1}_B]$.
  - No-signaling.
  - States are uniquely determined by probability assignments to pairs f_A, f_B.
- This does not give a unique tensor product, but a range of possibilities.
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Combining Systems: Tensor Products

Definition

Separable TP: $V(\Omega_A)^* \otimes_{\text{sep}} V(\Omega_B)^* =$
conv $\{\omega_A \otimes \omega_B | \omega_A \in V(\Omega_A)^*,\ \omega_B \in V(\Omega_B)^*\}$

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Maximal TP: $V(\Omega_A)^* \otimes_{\max} V(\Omega_B)^* = (V(\Omega_A) \otimes_{\text{sep}} V(\Omega_B))^*$

Definition

A tensor product $V(\Omega_A)^* \otimes V(\Omega_B)^*$ is a convex cone that satisfies

$$V(\Omega_A)^* \otimes_{\text{sep}} V(\Omega_B)^* \subseteq V(\Omega_A)^* \otimes V(\Omega_B)^* \subseteq V(\Omega_A)^* \otimes_{\max} V(\Omega_B)^*.$$

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Distinguishability

Definition

A set of states $\{\omega_1, \omega_2, \ldots, \omega_N\}$, $\omega_j \in \Omega$, is jointly distinguishable if $\exists$ an observable $(f_1, f_2, \ldots, f_N)$ s.t.

$$f_j(\omega_k) = \delta_{jk}.$$

Fact

The set of pure states of Ω is jointly distinguishable iff Ω is a simplex.

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Reduced States and Maps

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Given a state $v_{AB} \in V_A \otimes V_B$, the marginal state on V_A is defined by

$$\forall f_A \in V_A^*, \quad f_A(v_A) = f_A \otimes \tilde{1}_B(\omega_{AB}).$$

Definition

Given an affine map $\phi_{BC|A} : V_A \to V_B \otimes V_C$, the reduced map $\phi : V_A \to V_B$ is defined by

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A state $\omega \in \Omega$ is broadcast by a NPA map $\phi_{A'A''|A} : V_A \to V_{A'} \otimes V_{A''}$ if $\phi_{A'|A}(\omega) = \phi_{A''|A}(\omega) = \omega$.

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The No-Broadcasting Theorem

Theorem

A set of states is co-broadcastable iff it is contained in a simplex that has jointly distinguishable vertices.

- Quantum theory: states must commute.
- Universal broadcasting only possible in classical theories.

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The set of states broadcast by any affine map is a simplex that has jointly distinguishable vertices.

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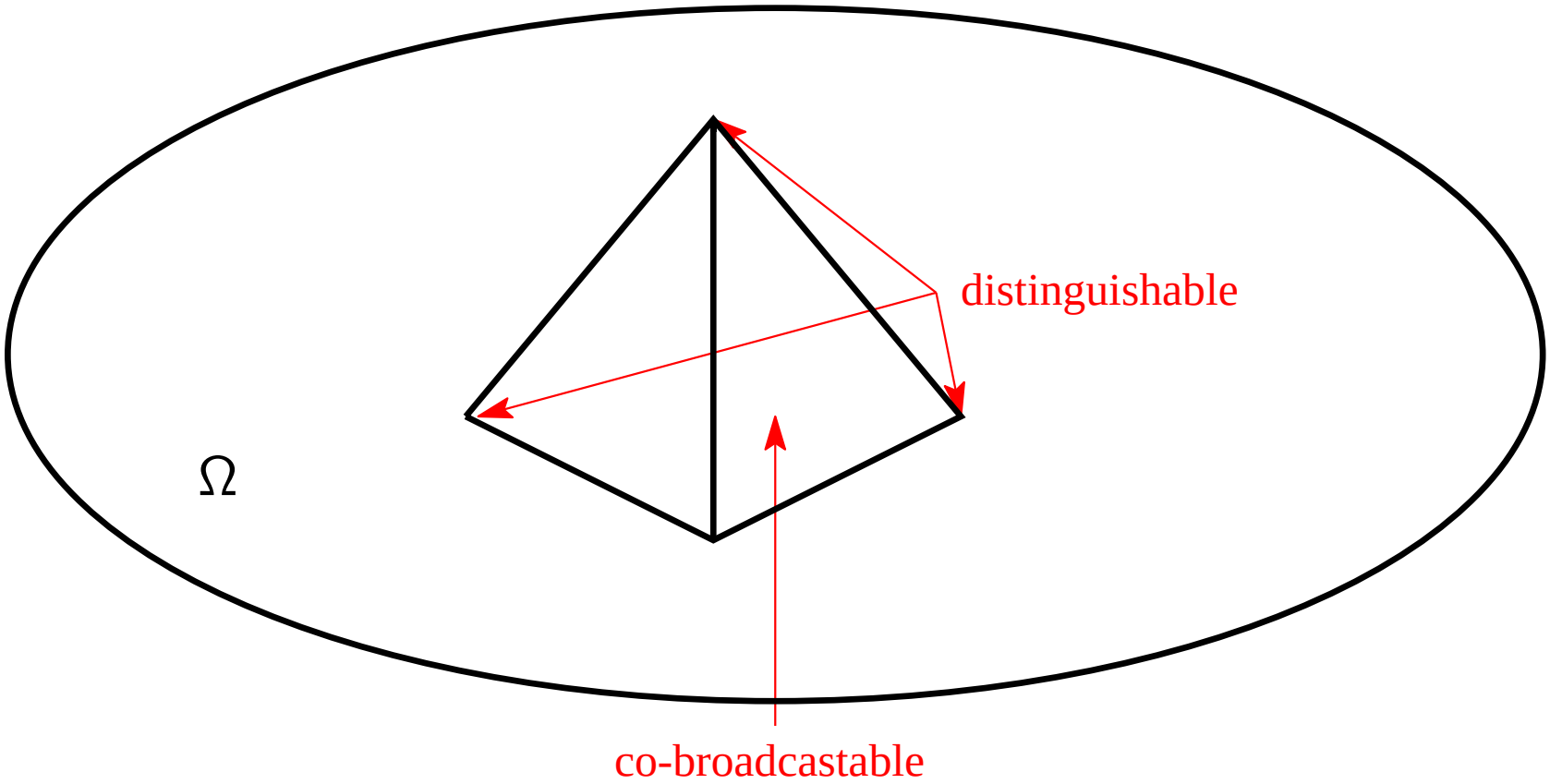

Comparison to CBH Theorem

Like CBH we have:

- No broadcasting $\Rightarrow$ State spaces of individual systems are nonclassical.
- No bit commitment $\Rightarrow$ Entangled states must exist.

Unlike CBH:

- No signaling has become a framework assumption.
- Postulates are genuinely independent.
- We are not particularly close to quantum theory.

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Conjecture

- Weak version: If the set of joint probabilities that Alice and Bob can obtain in a "prepare and measure" setup is the same as when those they can obtain from making measurements on joint states then bit commitment is impossible
- Strong version: In all other theories there is a secure bit commitment protocol.

Note:

- Applies to C^*-theories, i.e. classical and quantum, due to Choi-Jamiołkowski isomorphism, but it's weaker than this.
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