Quantum Causal Networks: A Quantum Informationish Approach to Causality in Quantum Theory

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Outline

Disclaimer!

The author of this presentation makes no claims as to the accuracy of any speculations about quantum gravity contained within. Such speculations do not necessarily represent the views of the author, or indeed any sane person, living or dead. Any similarity to existing formalisms for theories of quantum gravity is purely coincidental. The finiteness of all Hilbert spaces in this presentation does not reflect any views about the discreteness of spacetime. You can probably do everything with C^*-algebras if you prefer. This is a work in progress.

The Church of The Smaller Hilbert Space

Core Beliefs

- A preference for density operators, CP-maps and POVMs over state-vectors, unitary evolution and projective measurements.
  - The latter turn out to be fairly boring in the present framework.
- States belong to agents, not to systems.

"Other authors introduce a wave function for the whole Universe. In this book, I shall refrain from using concepts that I do not understand." Peres.

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Three lessons from Quantum Information

1. Quantum Theory is a noncommutative, operator-valued, generalization of probability theory.
2. We don't have to invoke gravity in order to encounter scenarios with unusual causal structures.
3. Causal ordering of events matters a lot less than you might think for describing quantum processes.

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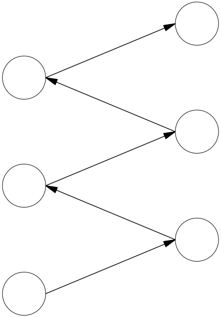
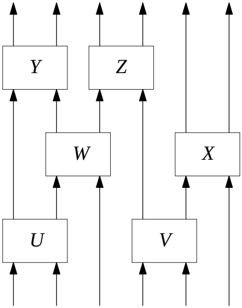
Unusual Causality without Gravity

Quantum Protocols



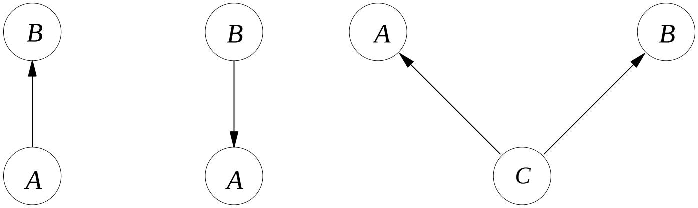
Unusual Causality without Gravity
Quantum Computing



- Quantum Circuits
- Measurement Based Quantum Computing

Irrelevance of Causal Ordering

The "small miracle" of time in quantum information



- In all three causal scenarios the predictions can be calculated from the formula $\mathrm{Tr}\left(M_A \otimes M_B \rho_{AB}\right)$.
- This follows from the Choi-Jamiołkowski isomorphism and is very useful in qinfo, e.g. crypto security proofs.
- c.f. P(X,Y) - I want a formalism in which it is that obvious.

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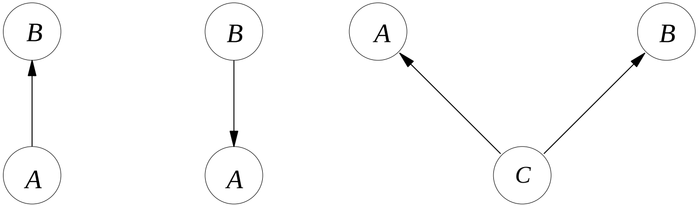

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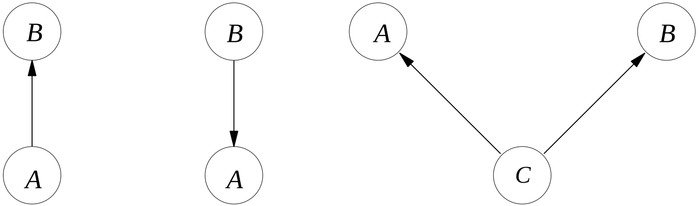

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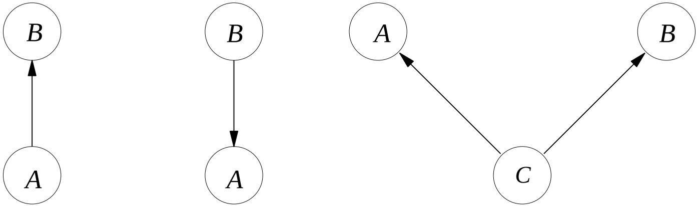

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Related Formalisms

- Hardy's Causaloid gr-qc/0509120, arXiv:gr-qc/0608043.
- Consistent/Decoherent Histories - particularly Isham's version quant-ph/9506028.
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Comparison to Quantum Causal Histories

- I don't require global unitarity.
- I only deal with finite causal structures.
- No free choice of initial conditions.
- No free entanglement in the initial state.

Note: Quantum Causal Histories could have been called Quantum Causal Networks or Quantum Bayesian Networks.

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Conditional Density Operators

Definition

A Conditional Density Operator (CDO) $\rho_{B|A} \in \mathfrak{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a positive operator that satisfies $\text{Tr}_B\left(\rho_{B|A}\right) = I_A$, where I_A is the identity operator on $\mathcal{H}_A$.

- c.f. $\sum_Y P(Y|X) = 1$
- Note: A density operator determines a CDO via $\rho_{B|A} = \rho_A^{-\frac{1}{2}} \rho_{AB} \rho_A^{-\frac{1}{2}}$.
- Notation: $M * N = M^{\frac{1}{2}} N M^{\frac{1}{2}}$
- $\rho_{B|A} = \rho_A^{-1} * \rho_{AB}$ and $\rho_{AB} = \rho_A * \rho_{AB}$.
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The Partial Transpose

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Given a basis $\{|j\rangle_A\}$ of $\mathcal{H}_A$ and an operator $M_{AB} = \sum_{jklm} M_{jk;lm} |jk\rangle \langle lm|_{AB} \in \mathfrak{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the partial transpose map on system A is given by

$$M_{AB}^{T_A} = \sum_{jklm} M_{jk;lm} |lk\rangle \langle jm|_{AB} \,. \tag{1}$$

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The Markov Evolution Law

Theorem

Let $\mathcal{E}_{B|A} : \mathfrak{L}(\mathcal{H}_A) \rightarrow \mathfrak{L}(\mathcal{H}_B)$ be a TPCP map and let be a density operator. Then for any $\rho_{AC} \in \mathfrak{L}(\mathcal{H}_A \otimes \mathcal{H}_C)$,

$$\mathcal{E}_{B|A} \otimes \mathcal{I}_C (\rho_{AC}) = Tr_A \left(\rho_{B|A} \rho_{AC} \right), \quad (2)$$

for some fixed CDO $\rho_{B|A}$.

- c.f. Classical stochastic dynamics $P(Y, Z) = \sum_X P(Y|X) P(X, Z)$.

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- This is just a restatement of the Choi-Jamiołkowski isomorphism.

$$\text{Let } \left|\Phi^+\right\rangle_{A'A} = \sum_j |jj\rangle_{A'A}.$$

$$\text{Then, } \rho_{B|A} = \mathcal{E}_{B|A'} \otimes \mathcal{I}_A \left(\left|\Phi^+\right\rangle\left\langle\Phi^+\right|_{A'A}\right)$$

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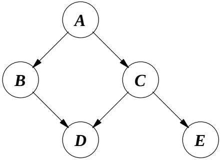
Directed Acyclic Graphs (DAGs)

- A DAG $G = (V, E)$ models causal structure.



- It is isomorphic to the Hasse diagram of a finite poset.
- Parents: $p(v) = \{u \in V : (u, v) \in E\}$.
- Children: $c(v) = \{u \in V : (v, u) \in E\}$.
- $p(D) = \{B, C\}$, $c(D) = \emptyset$,
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- Ancestral ordering, e.g. (A, B, C, D, E) or (A, C, E, B, D)

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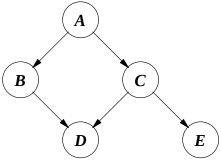

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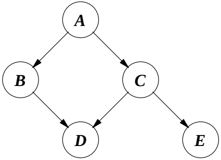

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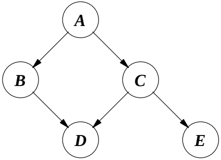

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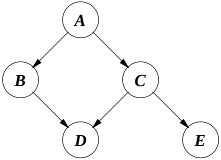
Classical Causal Networks

- A Classical Causal Network is a DAG $G = (V, E)$ together with a probability distribution $P(V)$ that factorizes according to

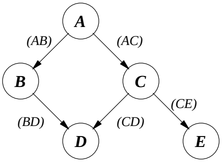
$$P(V) = \prod_{v \in V} P(v|p(v))$$



- $P(A, B, C, D, E) = P(A)P(B|A)P(C|A)P(D|B, C)P(E|C)$

Quantum Causal Networks

- In the quantum case we have to deal with no-cloning/no-broadcasting.
- We cannot set CDOs independently of each other.
- This can be solved by putting Hilbert spaces on the edges.



- Each vertex is associated with two TPCP maps
  - A fission isometry, e.g. $\mathcal{E}_{(CE)(CD)|C}$
  - A fusion map, e.g. $\mathcal{F}_{C|(AC)}$

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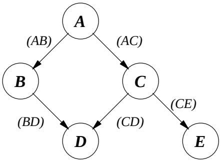

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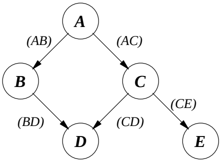

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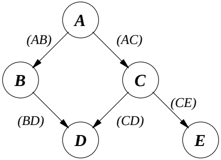
States on Quantum Causal Networks

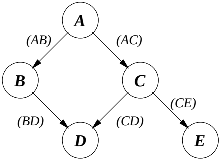
- The state on any "spacelike slice" can be found by composing fusion and fission maps.



- $\rho_A = \mathcal{F}_{A|\emptyset}(1)$
- $\rho_{BC} = \mathcal{F}_{B|(AB)} \otimes \mathcal{F}_{C|(AC)} \left(\mathcal{E}_{(AB)(AC)|A}\left(\rho_A\right)\right)$
- $\rho_{DE} = \mathcal{F}_{D|(BD)(CD)} \otimes \mathcal{F}_{E|(CE)} \left(\mathcal{E}_{(BD)|B} \otimes \mathcal{E}_{(CD)(CE)|C}\left(\rho_{BC}\right)\right)$
- The states are consistent on any "foliation".

CDOs to the rescue!

- This is a complete mess!
- Define CDOs $\rho_{v|p(v)}$ by combining fusion and fission maps.



- Let τ be the CDO associated with a fusion map $\mathcal{F}$.
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- $\rho_{D|BC} = \mathcal{E}^{\dagger}_{(BD)|B} \otimes \mathcal{E}^{\dagger}_{(CD)(CE)|C}\left(\tau_{D|(BD)(CD)} \otimes I_{(CE)}\right)$
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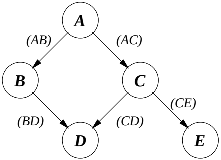

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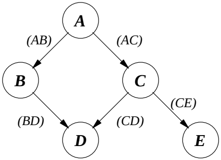
- $\rho_{D|BC} = \mathcal{E}^{\dagger}_{(BD)|B} \otimes \mathcal{E}^{\dagger}_{(CD)(CE)|C}\left(\tau_{D|(BD)(CD)} \otimes I_{(CE)}\right)$
- $\rho_{E|C} = \mathcal{E}^{\dagger}_{(CD)(CE)|C}\left(\tau_{E|(CE)}\right)$

CDOs to the rescue!

- This is a complete mess!
- Define CDOs $\rho_{v|p(v)}$ by combining fusion and fission maps.



- Let τ be the CDO associated with a fusion map $\mathcal{F}$.
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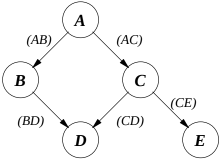
Decomposition into CDOs



- $\rho_{ABCDE} = (((\rho_A * \rho_{B|A}) * \rho_{C|A}) * \rho_{D|BC}) * \rho_{E|C}$ is a locally positive operator with the correct reduced states on all "spacelike slices".
- It doesn't depend on the choice of ancestral ordering: $\rho_{ABCDE} = (((\rho_A * \rho_{C|A}) * \rho_{E|C}) * \rho_{B|A}) * \rho_{D|BC}$

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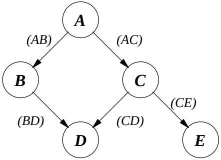

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Markov Equivalence

Because the result is so similar to the classical case, we can transcribe many known theorems with ease.

Definition

Two DAGs are Markov Equivalent if they support the same set of “density operators” (up to local positive maps).

Theorem

Two DAGs are Markov Equivalent if they have the same links and the same set of uncoupled head-to-head meetings.

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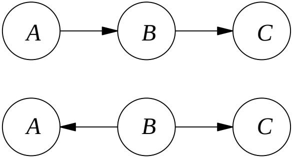
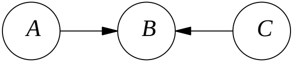
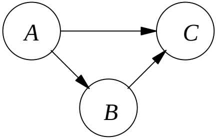


These are equivalent



Uncoupled head-to-head meeting



Coupled head-to-head meeting

Local Positivity: A Lament

- Life would be easier if it were $\rho_{B|A}$ rather than $\rho_{B|A}$.
- This stems from the fact that not all locally positive operators are positive and not all positive maps are completely positive.

$$\forall M_A, M_B \geq 0, \mathrm{Tr}\left(M_A \otimes M_B \rho_{AB}\right) \geq 0$$

$$\text{doesn't imply } \forall M_{AB} \geq 0, \mathrm{Tr}\left(M_{AB} \rho_{AB}\right) \geq 0$$

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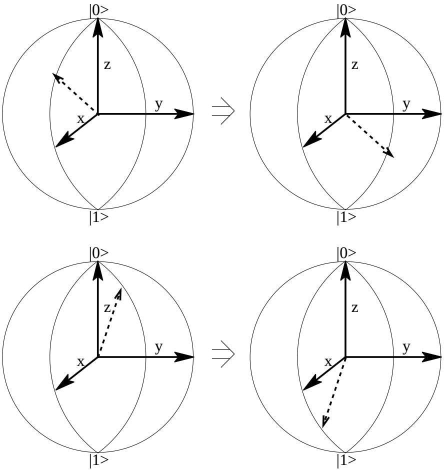
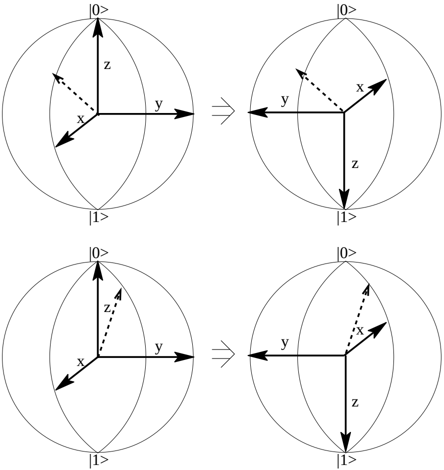
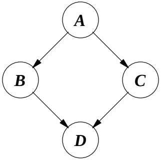
Example: Universal Not



Implemeting Universal Not Passively



So what is the problem?



- Whilst B and C are separate, we may perform any positive operation on them passively by just relabeling our POVM elements, obtaining any locally positive state ρ_{BC}.
- When we recombine them at D, we have to describe both systems in a common "reference frame".
- We must ensure that ρ_D is positive.

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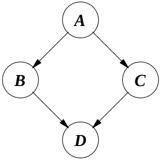

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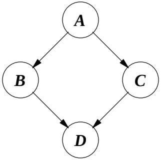

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Are you down with LPP?

Definition

A map $\mathcal{E}_{B_1B_2\ldots B_n|A_1A_2\ldots A_m} : \mathfrak{L}\left(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \ldots \otimes \mathcal{H}_{A_m}\right) \rightarrow \mathfrak{L}\left(\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \otimes \ldots \otimes \mathcal{H}_{B_n}\right)$ is completely local positivity preserving (CLPP) if

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is locally positive for all $\mathcal{H}_C$ and all locally positive operators $\rho_{A_1A_2\ldots A_mC}$.

- For $m = 1, n = 1$ CLPP = Positive.

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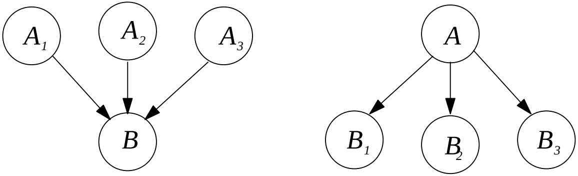
- For $m = 1, n = 1$ CLPP = Positive.

Are you down with LPP?

- I have no good classification of CLPP maps.
- It would be sufficient to classify CLPP maps of the form $\mathcal{E}_{B|A_1A_2\ldots A_m}$ and $\mathcal{E}_{B_1B_2\ldots B_n|A}$.



Lessons for Quantum Gravity?

- Correlation structure seems more intrinsic to quantum theory than causal structure.
- Maybe we don't have to sum over all possible causal structures - only Markov equivalence classes.

Open Questions

- Classification of CLPP maps.
- Including preparations and measurements - Quantum Influence Diagrams.

Acknowledgments

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