Simulation of 2-qubit Hamiltonians using general local operations

M. S. Leifer

September 2001

1 Introduction

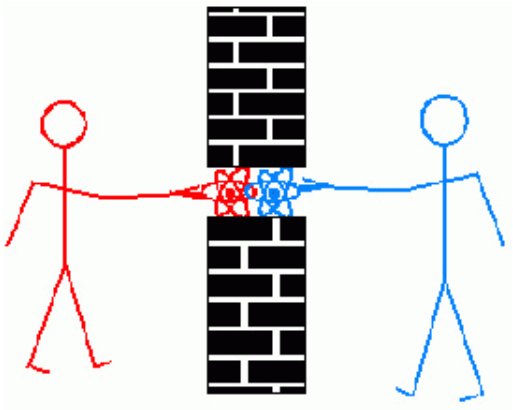

Figure 1: Alice and Bob, separated by a wall, through which they can only interact via the two qubit Hamiltonian H.[1]

Entanglement is a key rescource in many protocols in quantum information theory. One of the central problems of entanglement theory is to quantify and classify this rescource. Much work has been done on this for quantum states, but recently similar ideas have begun to be applied to quantum operations. The objective is to classify the ways in which operations can change the entanglement properties of the states they act on and to provide some measure of the amount of entangling interaction involved in the operation.

One of the main ways in which entanglement is quantified is by interconvertability of rescources under local operations. For example, in the problem considered here, Alice and Bob each have a qubit in a Hilbert space $\mathbb{C}^2$. Their combined system is a state in the 2-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$. However, as shown in fig. 1, they are separated by a wall and the only way their systems can interact is via a Hamiltonian H, which is always present. They would like to simulate the evolution that would occur if they had a different interaction Hamiltonian H', but they are only allowed to perform operations on their own systems.

In [3], we have found protocols for doing this with minimal time cost. We showed that the efficiency is not reduced if Alice and Bob are only allowed to peform unitary $U(2)$ operations on their systems, so this is the restricted case that is discussed here. We have solved the problem in terms of the geometry of efficiently simulable Hamiltonians and showed that the results can be succinctly characterised by a majorization like relation that provides a partial order on 2-qubit Hamiltonians.

2 The 2-qubit simulation problem

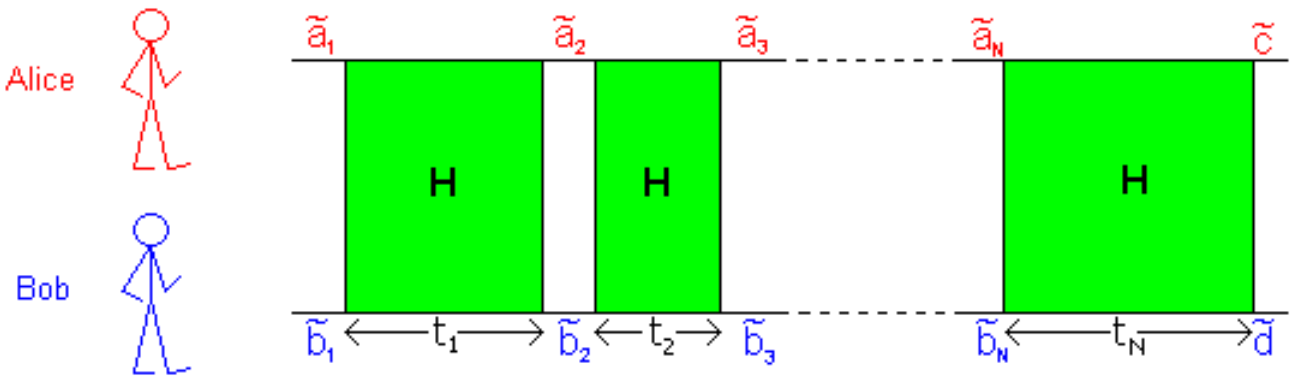

Figure 2: Protocol for two qubit Hamiltonian simulation. The resulting evolution will be $e^{-iH't'}$

If Alice and Bob were to do nothing for a time t then the evolution of their system due to H would be e^{-iHt}. Instead of this, we allow them to perform the protocol described below and shown in fig.2.

- Alice peforms operation $\tilde{a}_1 \in U(2)$ on her qubit and Bob performs $\tilde{b}_1 \in U(2)$ on his.
- They allow the system to evolve under H for time t_1.
- Alice peforms operation $\tilde{a}_2 \in U(2)$ on her qubit and Bob performs $\tilde{b}_2 \in U(2)$ on his.
- They allow the system to evolve under H for time t_2.

$\vdots$

- Alice peforms operation $\tilde{a}_N \in U(2)$ on her qubit and Bob performs $\tilde{b}_N \in U(2)$ on his.
- They allow the system to evolve under H for time t_N.
- Alice peforms operation $\tilde{c} \in U(2)$ on her qubit and Bob performs $\tilde{d} \in U(2)$ on his.

We assume that Alice and Bob can perform their local operations fast compared to the timescale of the interaction.

The protocol results in an evolution that would have occured if some Hamiltonian H' were acting for a time t'. Setting $t = \sum_{j=1}^{N} t_j$, we say that we have *simulated* the action of H' acting for time t' with efficiency t'/t. The resulting evolution is given by

$$\boxed{e^{-iH't'} = \left(\prod_{j=1}^{N} \tilde{a}_j \otimes \tilde{b}_j e^{-iHt_j} \right) \tilde{c} \otimes \tilde{d}} \qquad (1)$$

Additionaly, we require that the protocol must work with the same efficiency for all t, so that it simulates the dynamics due to H' at all intermediate times as well as resulting in $e^{-iH't'}$.

Definition 1 *H' is* efficiently simulable *by H ($H' \prec_S H$) if there exists a protocol with efficiency ≥ 1. If $H' \prec_S H$ and $H \prec_S H'$ then they are equivalent $H' \sim_S H$.*

We are interested in solving the following two equivalent problems.

- The efficient simulation problem: given H, find all $H' \prec_S H$.
- The optimal simulation problem: given H, H' find the maximal η such that $\eta H' \prec_S H$ and the protocol that achieves it. The maximal η is called the optimal simulation factor $\eta_{H|H'}$.

We have shown that every H is equivalent to its *normal form* which can be written in terms of three parameters $h_1 \geq h_2 \geq |h_3| \geq 0$.

$$\boxed{H \sim_S \sum_j h_j \sigma_j \otimes \sigma_j} \tag{2}$$

Thus, we only have to consider simulation of normal forms. We have also shown that eq. (1) can be rewritten as

$$\boxed{\eta D_{H'} = \sum_{j=1}^{N} p_j O_j D_H Q_j} \tag{3}$$

where $O_j, Q_j \in SO(3)$, $\eta = t'/t$ is the efficiency of the protocol, $\sum_j p_j = 1$ and

$$D_H = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}, D_{H'} = \begin{pmatrix} h'_1 & 0 & 0 \\ 0 & h'_2 & 0 \\ 0 & 0 & h'_3 \end{pmatrix} \tag{4}$$

In addition each term $O_j D_H Q_j$ can be taken to be diagonal. Thus, we can represent every Hamiltonian as a real 3-dimensional vector $h = (h_1, h_2, h_3)$ and determining the efficiently simulable $h' = (h'_1, h'_2, h'_3)$ becomes a problem in convex geometry.

3 The structure of $\mathcal{P}_H$

Since each term $O_j D_H Q_j$ in eq.(3) is diagonal, we can regard them as 3-dimensional vectors. The set of allowed $O_j D_H Q_j$ by contains at most 24 elements. Since the sum in eq.(3) is convex, the set of efficiently simulable Hamiltonians is a polytope, denoted $\mathcal{P}_H$, with these elements as its vertices. Fig. 3 shows $\mathcal{P}_H$ for the generic case where $h_1 > h_2 > h_3 > 0$.

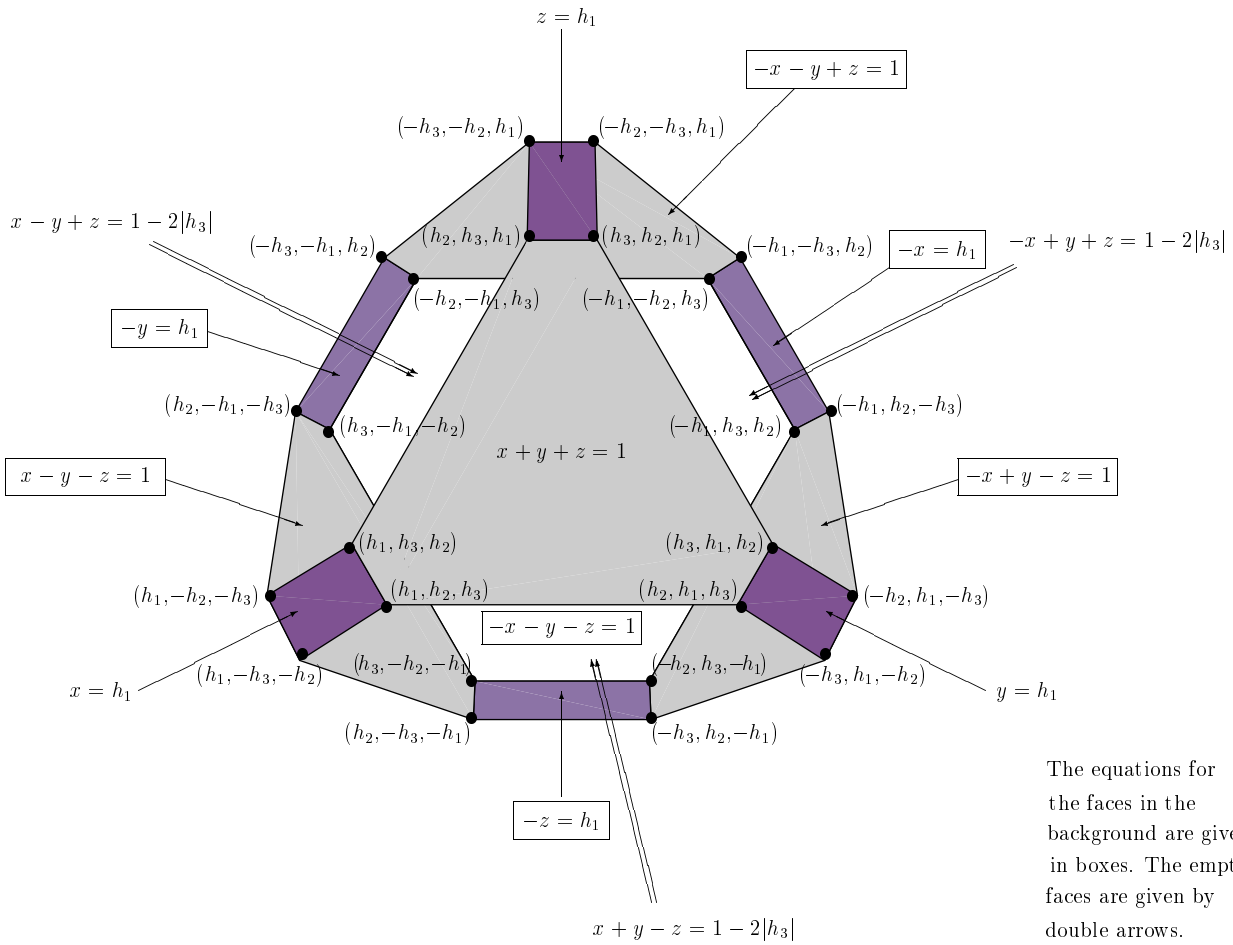

Figure 3: Structure of $\mathcal{P}_H$ [2]

The interior of $\mathcal{P}_H$ can be characterised by the following set of inequalities.

$$(x, y, z) \in \mathcal{P}_H \text{ iff } \begin{cases} |x| \leq h_1 \ , |y| \leq h_1 \ , |z| \leq h_1 \\ -(1 - 2h_3) \leq +x + y + z \leq 1 \\ -(1 - 2h_3) \leq -x - y + z \leq 1 \\ -(1 - 2h_3) \leq +x - y - z \leq 1 \\ -(1 - 2h_3) \leq -x + y - z \leq 1 \end{cases} \tag{5}$$

All Hamiltonians that lie within $\mathcal{P}_H$ are efficiently simulable. The optimal simulation problem can be solved by constructing the vector from the origin to the boundary of $\mathcal{P}_H$ that lies in the direction $h' = (h'_1, h'_2, h'_3)$. Detailed analysis of this can be found in [3].

4 The s-majorization relation

It is possible to rewrite our results in terms of a majorization like relation, which we call s-majorization (denoted $\prec_s$). Thus, H' is efficiently simulable using H iff $h' \prec_s h$. In terms of components, this means

$$\begin{array}{lcl} h'_1 & \leq & h_1 \\ h'_1 + h'_2 - h'_3 & \leq & h_1 + h_2 - h_3 \\ h'_1 + h'_2 + h'_3 & \leq & h_1 + h_2 + h_3 \end{array} \tag{6}$$

Comments

- s-majorization is a partial order on the space of 2-qubit Hamiltonians.
- s-majorization implies the standard weak sub-majorization relation ($h' \prec_w h$). Therefore, weak sub-majorization is a necessary, but not sufficient condition for efficient simulation.
- In the case where $\mathrm{sg}(h'_3) = \mathrm{sg}(h_3)$ and $h'_1 + h'_2 + h'_3 = h_1 + h_2 + h_3$, majorization, weak sub-majorization and s-majorization are all equivalent.
- The optimal simulation factor is given by $\eta_{H|H'} = \max_\eta(\eta H' \prec_s H)$.

5 Conclusions

We have found the optimal simulation protocols for 2-qubit Hamiltonians and shown that they lead to a partial order, which is similar to a majorization relation. This provides a good starting point for the classification of entanglement in operations. A natural next step would be to generalise this to higher dimensional systems and to more than two parties. Some work has been done on this and a few bounds are known.

Also, we would like to to connect this work to the case where we want to simulate a particular unitary, but we don't require that the entire dynamics of H' is simulated. Another interesting direction would be to investigate the robustness of these protocols to uncertainties in the Hamiltonian and the local operations. Other related questions include quantifying the classical communication capacity of a Hamiltonian and the maximum possible entanglement obtainable from an operation. Finally, generalisations to the case where we have more than one copy of the Hamiltonian acting on different systems might provide new measures of entanglement for operations.

References

[1] Diagram due to C. H. Bennett.

[2] Diagram due to D. Leung.

[3] C. H. Bennett, J. I. Cirac, M. S. Leifer, D. W. Leung, N. Linden, S. Popescu, and G. Vidal. Optimal simulation of two-qubit hamiltonians using general local operations. quant-ph/0107035, 2001.