Quantum Dynamics as Generalized Conditional Probabilities

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Quantum Theory as Generalized Probability

Classical	Quantum
Algebra of R.V.'s on a sample space (Ω_A, S_A).	Algebra of operators $\mathfrak{B}(\mathcal{H}_A)$ on a Hilbert space $\mathcal{H}_A$.
Probability distribution: $P(X)$	Quantum State: $\rho_A \in \mathfrak{B}(\mathcal{H}_A)$
Expectation value: $\langle X \rangle = \sum_X X P(X)$	Expectation value: $\langle X \rangle = \mathrm{Tr}\,(X \rho_A)$
Cartesian Product: $\Omega_{AB} = \Omega_A \times \Omega_B$	Tensor Product: $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$
Joint Distribution: $P(X, Y)$	Joint state: $\rho_{AB} \in \mathfrak{B}(\mathcal{H}_{AB})$
Marginal Distn.: $P(X) = \sum_Y P(X, Y)$	Partial Trace: $\rho_A = \mathrm{Tr}_B\,(\rho_{AB})$
Stochastic transition map: $\Gamma_{Y\|X}$	TPCP map $\mathcal{E}_{B\|A} : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_B)$
Conditional Probability: $P(Y\|X) = \frac{P(X, Y)}{P(X)}$	?

Generalized Probability Theory

* Quantum theory and classical probability are part of a more general theory, with $\mathfrak{B}(\mathcal{H})$ replaced by a more general C^*algebra.

* In this talk, specialize to finite dimensional algebras of the form:

$$\mathfrak{O} = \mathfrak{B}\left(\mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2} \oplus \ldots \oplus \mathbb{C}^{d_N}\right)$$

* We are mainly interested in these two special cases:
    * Classical probability with a finite sample space

$$\mathfrak{O}_{c,d} = \mathfrak{B}(\mathbb{C}^{\oplus d}) = \mathfrak{B}(\mathbb{C} \oplus \mathbb{C} \oplus \ldots \oplus \mathbb{C})$$

    * "Full" finite-dimensional quantum theory

$$\mathfrak{O}_{q,d} = \mathfrak{B}(\mathbb{C}^d)$$

Why quantum conditional probability?

* "Practical" Reasons:
  * Several probabilistic structures require cond. prob. or cond. independence for their definition, e.g. Markov Chains, Bayesian Networks.
  * Better understand the relationships between different qinfo tasks.
* "Foundational" reasons:
  * QT is actually more like a generalized theory of stochastic process than abstract Kolmogorov probability. Spacelike and timelike events are combined differently.
  * Cond. prob. is the missing notion that relates the two.
  * Could be relevant to applying QT in the absence of background causal structure.

Outline

1. Introduction
2. Stochastic Dynamics as Conditional Probability
3. Choi-Jamiolkowski Isomorphism
4. A New Isomorphism
5. Operational Interpretation
6. Application: Cloning, broadcasting & monogamy of entanglement
7. Future Directions

1. Introduction: Uses of Cond. Prob.

(A) Reconstructing a joint distribution from a marginal

$$P(X,Y) = P(Y|X)P(X)$$

(B) Bayesian Updating

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

(C) Stochastic Dynamics

$$P(Y) = \sum_X \Gamma_{Y|X} P(X)$$

(D) Conditional Shannon Entropy

$$H(Y|X) = H(X,Y) - H(X) = -\sum_{X,Y} P(X,Y) \log_2 P(Y|X)$$

(E) Reduction of complexity via conditional independence

$$P(Z|X,Y) = P(Z|X) \Leftrightarrow P(X,Y,Z) = P(X)P(Y|X)P(Z|X)$$

1. Introduction: Uses of Cond. Prob.

(A) Reconstructing a joint state from a marginal

$$\rho_{AB} = f(\rho_A, \rho_{B|A})$$

(B) Updating a state after a measurement

$$\rho_A \to \frac{\mathcal{E}^{(j)}\left(\rho_A\right)}{\mathrm{Tr}\left(\mathcal{E}^{(j)}\left(\rho_A\right)\right)}$$

(C) TPCP Dynamics $\quad \rho_B = \mathcal{E}_{B|A}(\rho_A)$

(D) Conditional von Neumann Entropy

$$S(B|A) = -\mathrm{Tr}\left(\rho_{AB}\log_2 \rho_{B|A}\right)$$

(E) Reduction of complexity via conditional independence

* Cerf & Adami achieve (A), (D), (E). I achieve (A), (B), (C).

1. Introduction: Cerf & Adami

* Cerf & Adami: Let $\rho_{AB} \in \mathfrak{D}_A \otimes \mathfrak{D}_B$ be a density operator, and define

$$\rho_{B|A} = \lim_{n\to\infty} \left[(\rho_A \otimes I_B)^{-\frac{1}{2n}} \rho_{AB}^{\frac{1}{n}} (\rho_A \otimes I_B)^{-\frac{1}{2n}} \right]^n$$

$$\log_2 \rho_{B|A} = \log_2 \rho_{AB} - \log_2 \rho_A \otimes I_B$$

* (A) Reconstruction: $\rho_{AB} = 2^{\log_2 \rho_A \otimes I_B + \log_2 \rho_{B|A}}$
* (C) Entropy:

$$S(B|A) = S(A,B) - S(A) = -\mathrm{Tr}\left(\rho_{AB} \log_2 \rho_{B|A}\right)$$

* (E) Complexity Reduction:

$$\log_2 \rho_{C|AB} = \rho_{C|A} \otimes I_B \Leftrightarrow \log_2 \rho_{ABC} = \log_2 \rho_A \otimes I_B \otimes I_C + \log_2 \rho_{B|A} \otimes I_C + \log_2 \rho_{C|A} \otimes I_B$$

$$S(B:C|A) = S(C|A) - S(C|A,B) = 0$$

1. Introduction: An Obvious Alternative

* Let $\rho_{AB} \in \mathfrak{H}_A \otimes \mathfrak{H}_B$ be a density operator, and define:

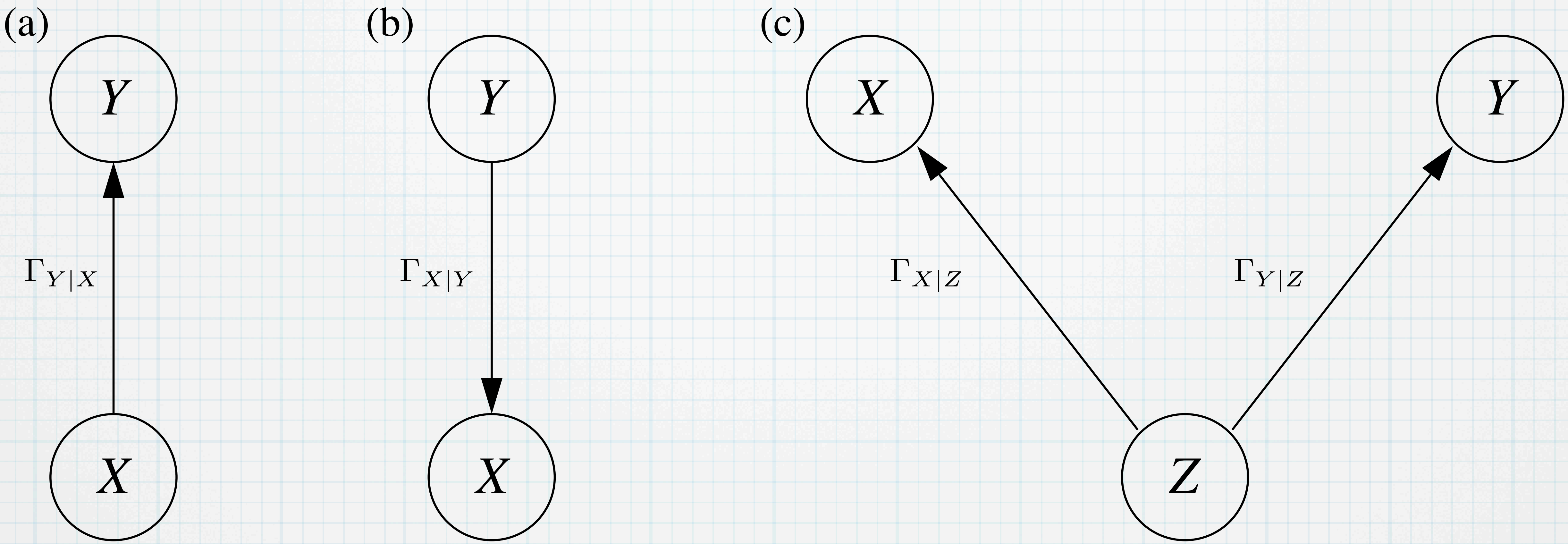
$$\rho_{B|A} = \rho_A^{-\frac{1}{2}} \otimes I_B \rho_{AB} \rho_A^{-\frac{1}{2}} \otimes I_B$$

* Properties:
  * $\rho_{B|A}$ is a density operator on $\mathfrak{H}_A \otimes \mathfrak{H}_B$.
  * Maximally mixed on subsystem A: $\mathrm{Tr}_B(\rho_{B|A}) = \frac{P_A}{d_A^r}$.

* (A) Reconstruction:

$$\rho_{AB} = \rho_A^{\frac{1}{2}} \otimes I_B \rho_{AB} \rho_A^{\frac{1}{2}} \otimes I_B$$

2. Dynamics as Conditional Probability



$$P(Y) = \sum_X \Gamma_{Y|X} P(X)$$

$$P(X,Y) = \Gamma_{Y|X} P(X)$$

$$P(Y|X) = \Gamma^r_{Y|X}$$

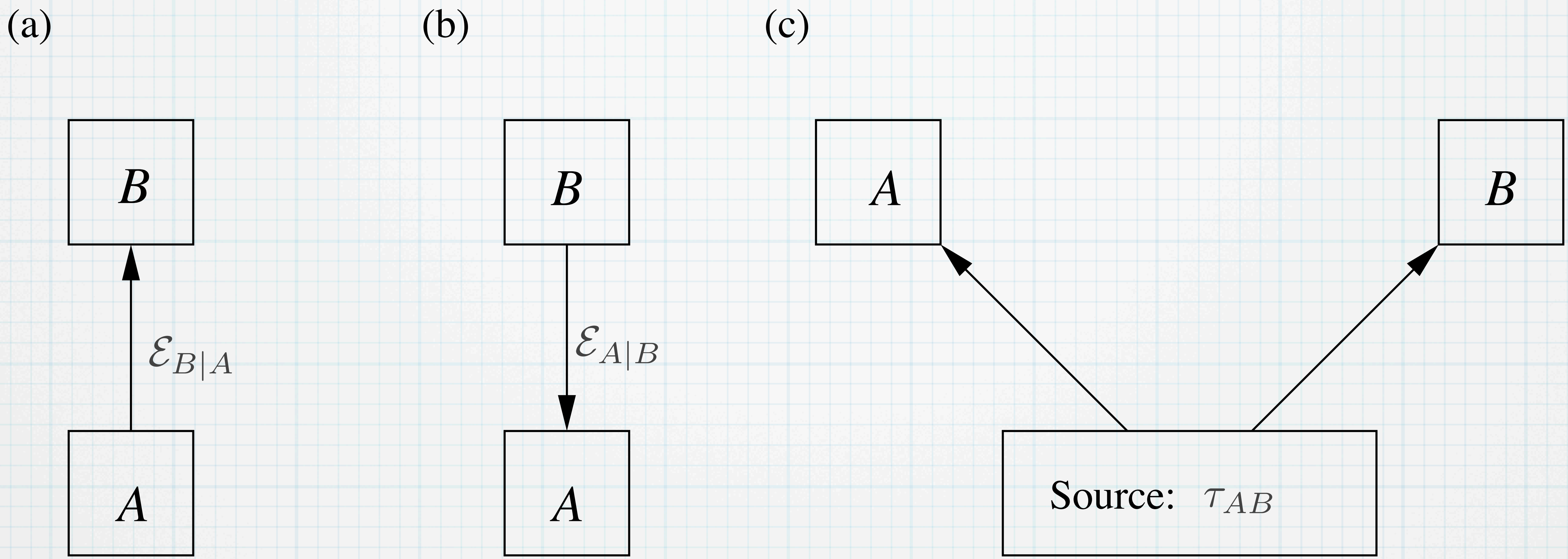
$$P(X,Y)$$

$$P(X) = \sum_Y P(X,Y)$$

$$P(Y|X) = \frac{P(X,Y)}{P(X)}$$

Isomorphism: $\left(P(X), \Gamma^r_{Y|X}\right) \Leftrightarrow P(X,Y)$

2. Dynamics as conditional probability



$\rho_B = \mathcal{E}_{B|A}(\rho_A)$

$\rho_{AB} = ?$

$\rho_{B|A} = ?$

τ_{AB}

$\tau_A = \mathrm{Tr}_B\left(\tau_{AB}\right)$

$\tau_{B|A} = ?$

Isomorphism: $\left(\rho_A, \mathcal{E}^r_{B|A}\right) \Leftrightarrow \tau_{AB}$?

3. Choi-Jamiolkowski Isomorphism

* Recall: Kraus decomposition of CP-maps $\mathcal{E}_{B|A} : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_B)$

$$\mathcal{E}_{B|A}(\rho_A) = \sum_\mu R^{(\mu)}_{B|A} \rho_A R^{(\mu)\dagger}_{B|A} \qquad R^{(\mu)}_{B|A} : \mathcal{H}_A \to \mathcal{H}_B$$

* For bipartite pure states and operators:

$$R_{B|A} = \sum_{jk} \alpha_{jk} \left|j\right\rangle_B \left\langle k\right|_A \Leftrightarrow \left|\Psi\right\rangle_{AB} = \sum_{jk} \alpha_{jk} \left|k\right\rangle_A \otimes \left|j\right\rangle_B$$

* For mixed states and CP-maps:

$$R^{(\mu)}_{B|A} \Leftrightarrow \left|\Psi^{(\mu)}\right\rangle \qquad \mathcal{E}_{B|A} \Leftrightarrow \tau_{AB} = \sum_\mu \left|\Psi^{(\mu)}\right\rangle_{AB} \left\langle\Psi^{(\mu)}\right|_{AB}$$

3. Choi-Jamiolkowski Isomorphism

* Let $\left|\Phi^{+}\right\rangle_{AA'} = \frac{1}{\sqrt{d_A}} \sum_j \left|j\right\rangle_A \otimes \left|j\right\rangle_{A'}$

* Then $\tau_{AB} = \mathcal{I}_A \otimes \mathcal{E}_{B|A'} \left(\left|\Phi^{+}\right\rangle_{AA'} \left\langle\Phi^{+}\right|_{AA'}\right)$

$$\mathcal{E}_{B|A}\left(\rho_A\right) = d_A^2 \left\langle\Phi^{+}\right|_{AA'} \rho_A \otimes \tau_{A'B} \left|\Phi^{+}\right\rangle_{AA'}$$

* Operational interpretation: Noisy gate teleportation.

3. Choi-Jamiolkowski Isomorphism

* Remarks:
  * Isomorphism is basis dependent. A basis must be chosen to define $|\Phi^+\rangle_{AA'}$.
  * If we restrict attention to Trace Preserving CP-maps then

$$\tau_A = \mathrm{Tr}_B(\tau_{AB}) = \frac{I_A}{d_A}$$

  * This is a special case of the isomorphism we want to construct

$$\left(\rho_A, \mathcal{E}^r_{B|A}\right) \Leftrightarrow \tau_{AB}$$

where $\rho_A = \dfrac{I_A}{d_A}$.

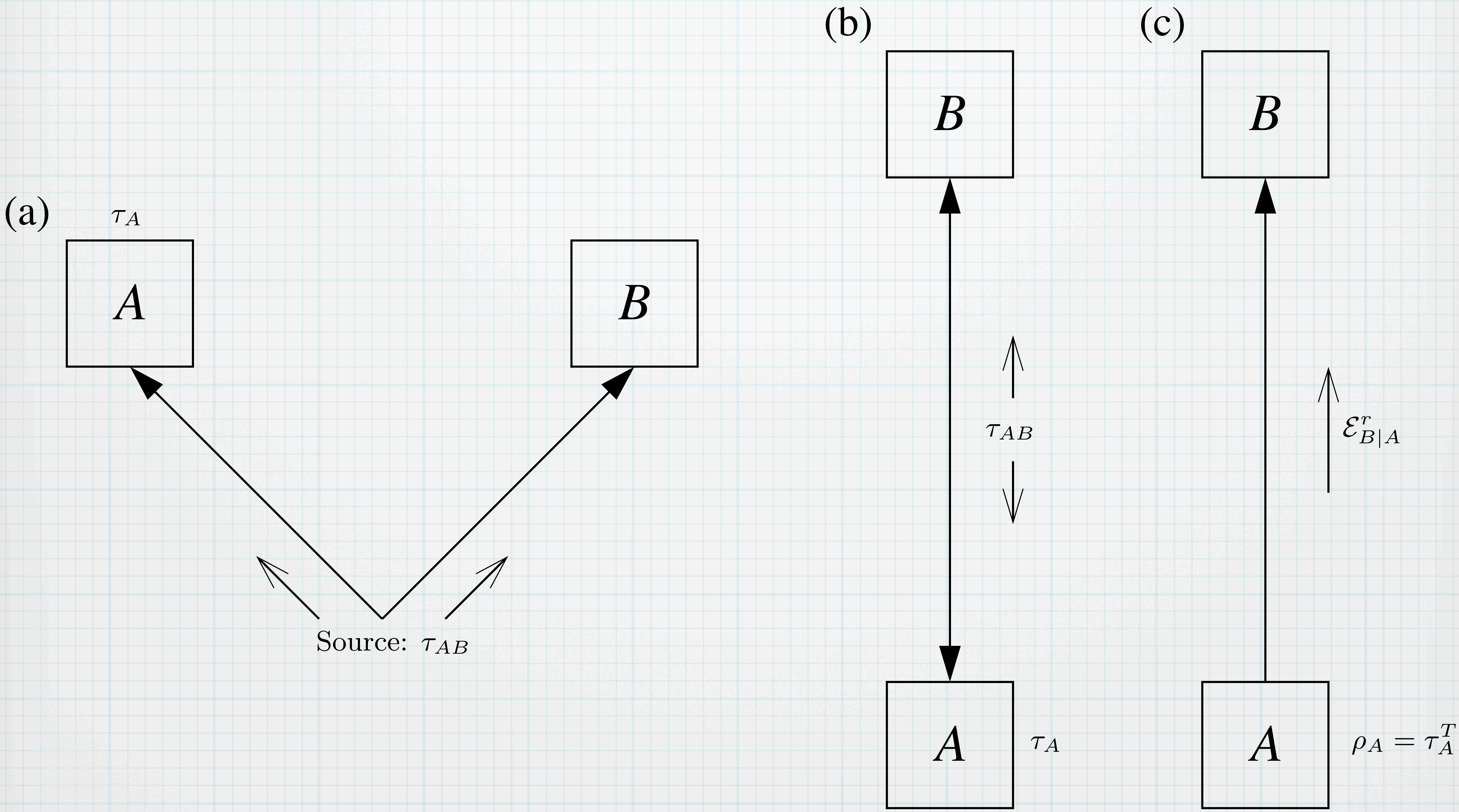
4. A New Isomorphism

* $\left(\rho_A, \mathcal{E}^r_{B|A}\right) \to \tau_{AB}$ direction:
  * Instead of $\left|\Phi^+\right\rangle_{AA'}$ use $\left|\Phi\right\rangle_{AA'} = \left(\rho_A^T\right)^{\frac{1}{2}} \otimes I_{A'} \left|\Phi^+\right\rangle_{AA'}$
  * Then $\tau_{AB} = \mathcal{I}_A \otimes \mathcal{E}^r_{B|A'} \left(\left|\Phi\right\rangle_{AA'} \left\langle\Phi\right|_{AA'}\right)$

* $\tau_{AB} \to \left(\rho_A, \mathcal{E}^r_{B|A}\right)$ direction:
  * Set $\rho_A = \tau_A^T, \qquad \tau_A = \mathrm{Tr}_B\left(\tau_{AB}\right)$
  * Let $\tau_{B|A} = \tau_A^{-\frac{1}{2}} \otimes I_B \tau_{AB} \tau_A^{-\frac{1}{2}} \otimes I_B$
  * $\tau_{B|A}$ satisfies $\mathrm{Tr}_B\left(\tau_{B|A}\right) = \dfrac{P_A}{d_A^r}$
  * Hence, it is uniquely associated to a TPCP map via the Choi-Jamiolkowski isomorphism.

$$\mathcal{E}^r_{B|A} : \mathfrak{B}(P_A \mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_B)$$

4. A New Isomorphism



5. Operational Interpretation

* Reminder about measurements:
  * POVM: $\mathcal{M} = \{\boldsymbol{M}\}, \quad \boldsymbol{M} > 0, \quad \sum_{\boldsymbol{M}} \boldsymbol{M} = I$
  * Probability Rule: $P(M) = \mathrm{Tr}\left(\boldsymbol{M}\rho\right)$
  * Update CP-map: $\rho_{|M} = \dfrac{\mathcal{E}^M(\rho)}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$

$$\mathcal{E}^M(\rho) = \sum_j A_j^M \rho A_j^{M\dagger} \qquad\qquad \sum_j A_j^{M\dagger} A_j^M = \boldsymbol{M}$$

  * $\mathcal{E}^M$ depends on details of system-measuring device interaction.

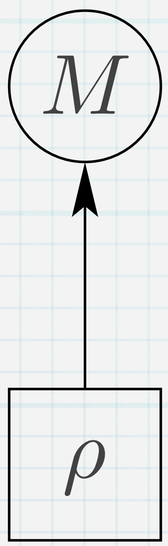
5. Operational Interpretation

* **Lemma:** $\rho = \sum_M P(M)\rho_{|M}$ is an ensemble decomposition of a density matrix ρ iff there is a POVM $M = \{\boldsymbol{M}\}$ s.t.

$$P(M) = \mathrm{Tr}\left(\boldsymbol{M}\rho\right) \quad \text{and} \quad \rho_{|M} = \frac{\sqrt{\rho}\boldsymbol{M}\sqrt{\rho}}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$

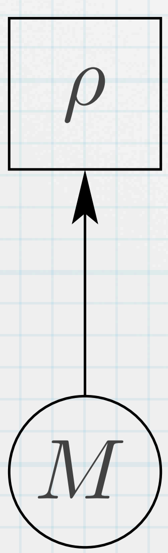
* **Proof sketch:** Set $\boldsymbol{M} = P(M)\rho^{-\frac{1}{2}}\rho_{|M}\rho^{-\frac{1}{2}}$

5. Operational Interpretation



* M-measurement of ρ
  * Input: ρ
  * Measurement probabilities: $P(M) = \mathrm{Tr}\left(\boldsymbol{M}\rho\right)$
  * Updated state:

$$\rho_{|M} = \frac{\sqrt{\boldsymbol{M}}\rho\sqrt{\boldsymbol{M}}}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$



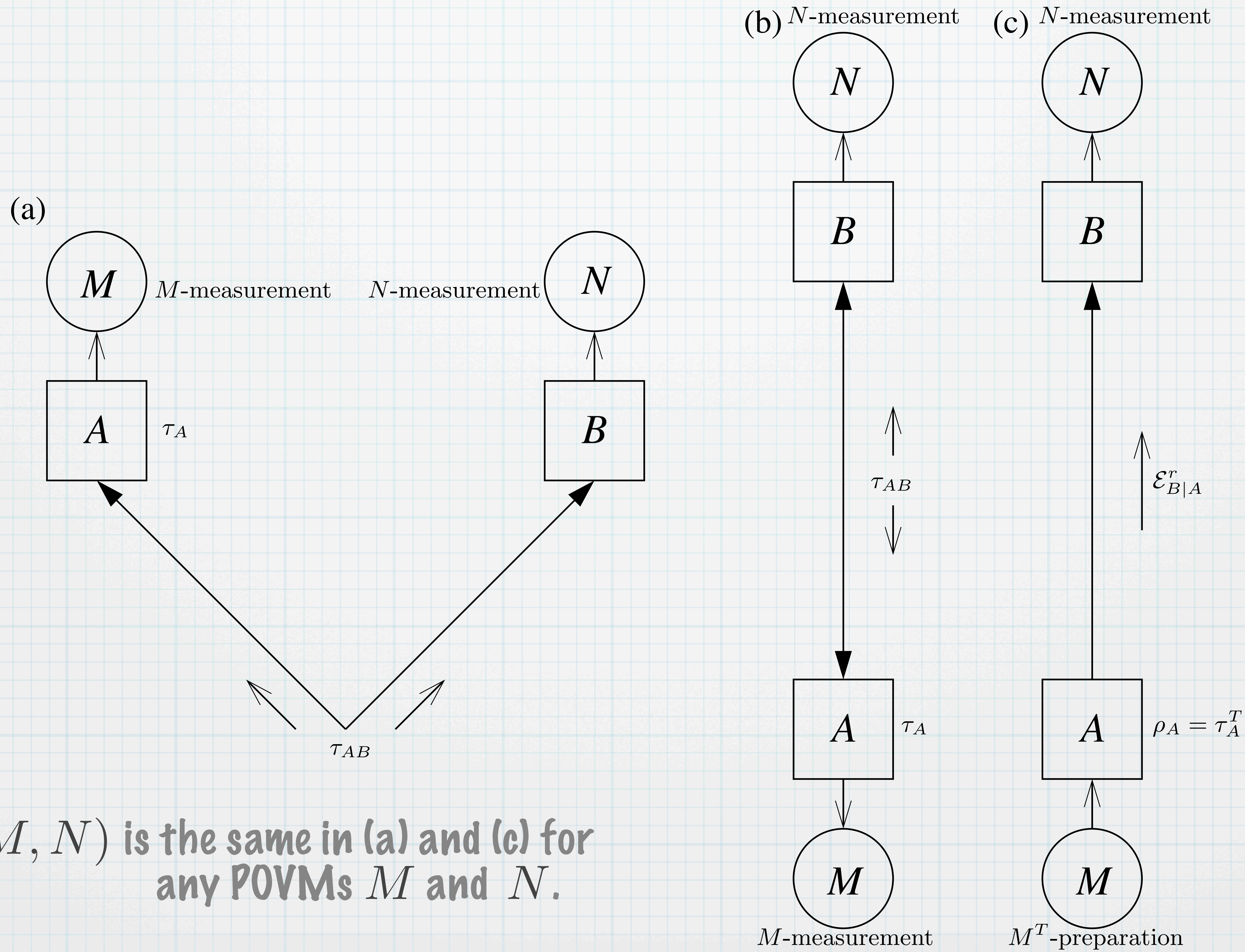
* M-preparation of ρ
  * Input: Generate a classical r.v. with p.d.f

$$P(M) = \mathrm{Tr}\left(\boldsymbol{M}\rho\right)$$

  * Prepare the corresponding state:

$$\rho_{|M} = \frac{\sqrt{\rho}\boldsymbol{M}\sqrt{\rho}}{\mathrm{Tr}\left(\boldsymbol{M}\rho\right)}$$

5. Operational Interpretation



$P(M, N)$ is the same in (a) and (c) for any POVMs M and N.

6. Application: Broadcasting & Monogamy

* For any TPCP map $\mathcal{E}_{BC|A} : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_A \otimes \mathcal{H}_C)$ the reduced maps are:

$$\mathcal{E}_{B|A} = \mathrm{Tr}_C \circ \mathcal{E}_{BC|A} \qquad \mathcal{E}_{C|A} = \mathrm{Tr}_B \circ \mathcal{E}_{BC|A}$$

* The following commutativity properties hold:

$$\begin{tikzcd}
\rho_{ABC} \arrow[r, equal] \arrow[d, "\mathrm{Tr}_C"'] & (\rho_A, \mathcal{E}^r_{BC|A}) \arrow[d, "\mathrm{Tr}_C"] \\
\rho_{AB} \arrow[r, equal] & (\rho_A, \mathcal{E}^r_{B|A}).
\end{tikzcd}$$

* Therefore, 2 states ρ_{AB}, ρ_{AC} incompatible with being the reduced states of a global state ρ_{ABC}.

* 2 reduced maps $\mathcal{E}^r_{B|A}, \mathcal{E}^r_{C|A}$ incompatible with being the reduced maps of a global map $\mathcal{E}^r_{BC|A}$.

6. Application: Broadcasting & Monogamy

- A TPCP-map $\mathcal{E}_{A'A''|A} : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_{A'} \otimes \mathcal{H}_{A''})$ is broadcasting for a state ρ_A if

$$\mathcal{E}_{A'|A}(\rho_A) = \rho_{A'} \qquad \mathcal{E}_{A''|A}(\rho_A) = \rho_{A''}$$

- A TPCP-map $\mathcal{E}_{A'A''|A} : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_{A'} \otimes \mathcal{H}_{A''})$ is cloning for a state ρ_A if

$$\mathcal{E}_{A'A''|A}(\rho_A) = \rho_{A'} \otimes \rho_{A''}$$

- Note: For pure states cloning = broadcasting.
- A TPCP-map is universal broadcasting if it is broadcasting for every state.

6. Application: Broadcasting & Monogamy

- No cloning theorem (Dieks '82, Wootters & Zurek '82):
  - There is no map that is cloning for two nonorthogonal and nonidentical pure states.
- No broadcasting theorem (Barnum et. al. '96):
  - There is no map that is broadcasting for two noncommuting density operators.
- Clearly, this implies no universal broadcasting as well.
- Note that the maps $\mathcal{E}_{A'|A}, \mathcal{E}_{A''|A}$ are valid individually, but they cannot be the reduced maps of a global map $\mathcal{E}_{A'A''|A}$.

6. Application: Broadcasting & Monogamy

* The maps $\mathcal{E}_{A'|A}, \mathcal{E}_{A''|A}$ must be related to incompatible states $\tau_{AA'}, \tau_{AA''}$

* Theorem: If $\mathcal{E}_{A'A''|A}$ is universal broadcasting, then both $\tau_{AA'}, \tau_{AA''}$ must be pure and maximally entangled.

* Ensemble broadcasting $\{(p,\rho_1),((1-p),\rho_2)\}$ s.t. $[\rho_1,\rho_2]\neq 0$

$$\left(p\rho_1+(1-p)\rho_2, \mathcal{E}^r_{A'A''|A}\right) \Leftrightarrow \tau_{AA'A''}$$

* Theorem: There is a local operation on A that transforms both $\tau_{AA'}$ and $\tau_{AA''}$ into pure, entangled states with nonzero probability of success.

7. Future Directions

* Quantitative relations between approximate ensemble broadcasting and monogamy inequalities for entanglement.
* Dynamics of systems initially correlated with the environment.
* Quantum pooling (joint work with R. Spekkens).

* How are the different analogs of conditional probability related?
* Is their a heirarchy of conditional independence relations?
* Can quantum theory be formulated using an analog of conditional probability as the fundamental notion?