Conditional Density Operators in Quantum Information

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Banff (12th February 2007)

quant-ph/0606022 Phys. Rev. A 74, 042310(2006)
quant-ph/0611233

Classical Conditional Probability

$$(\Omega, S, \mu) \qquad A, B \in S \qquad \mu(A) \neq 0$$

$$\mathrm{Prob}(B|A) = \frac{\mu\left(A \cap B\right)}{\mu\left(A\right)}$$

* Generally, A and B can be ANY events in the sample space.
* Special Cases:

1. A is hypothesis, B is observed data - Bayesian Updating
2. A and B refer to properties of two distinct systems
3. A and B refer to properties of a system at 2 different times - Stochastic Dynamics

* 2. and 3. - $\Omega = \Omega_1 \times \Omega_2$

Quantum Conditional Probability

* In Quantum Theory
  * $\Omega \rightarrow \mathcal{H}$ Hilbert Space
  * $S \rightarrow \{\text{closed s-spaces of } \mathcal{H}\}$
  * $\mu \rightarrow \rho$ Density operator
* What is the analog of $\text{Prob}(B|A)$?
* Definition should ideally encompass:
  1. Conditioning on classical data (measurement-update)
  2. Correlations between 2 subsystems w.r.t tensor product
  3. Correlations between same system at 2 times (TPCP maps)
  4. Correlations between ARBITRARY events, e.g. incompatible observables
* AND BE OPERATIONALLY MEANINGFUL!!!!!

Outline

1. Conditional Density Operator
2. Remarks on Conditional Independence
3. Choi-Jamiolkowski Revisited
4. Applications
5. Open Questions

1. Conditional Density Operator

* Classical case: Special cases 2. and 3. - $\Omega = \Omega_1 \times \Omega_2$
  * We'll generally assume s-spaces finite and deal with them by defining integer-valued random variables

$$\Omega_1 = \{X = j\}_{j \in \{1,2,\ldots N\}} \qquad \Omega_2 = \{Y = k\}_{k \in \{1,2,\ldots M\}}$$

  * $P(X = j, Y = k)$ abbreviated to $P(X, Y)$
  * $\sum_j f(P(X = j, Y = k))$ abbreviated to $\sum_X f(P(X, Y))$
  * Marginal $P(X) = \sum_Y P(X, Y)$
  * Conditional $P(Y|X) = \dfrac{P(X, Y)}{P(X)}$

1. Conditional Density Operator

* Can easily embed this into QM case 2, i.e. w.r.t. a tensor product $\mathcal{H}_X \otimes \mathcal{H}_Y$
    * Write

$$\rho_{XY} = \sum_{jk} P(X = j, Y = k) \left|j\right\rangle \left\langle j\right|_X \otimes \left|k\right\rangle \left\langle k\right|_Y$$

    * Conditional Density operator

$$\rho_{Y|X} = \sum_{jk} P(Y = k|X = j) \left|j\right\rangle \left\langle j\right|_X \otimes \left|k\right\rangle \left\langle k\right|_Y$$

    * Equivalently $\rho_{Y|X} = \rho_X^{-1} \otimes I_Y \rho_{XY}$
        * where $\rho_X = \mathrm{Tr}_Y \left(\rho_{XY}\right)$ and ρ_X^{-1} is the Moore-Penrose pseudoinverse.
    * More generally, $[\rho_X \otimes I_Y, \rho_{XY}] \neq 0$ so how to generalize the conditional density operator?

1. Conditional Density Operator

* A "reasonable" family of generalizations is:

$$\rho_{Y|X}^{(n)} = \left(\rho_X^{-\frac{1}{2n}} \otimes I_Y \rho_{XY}^{\frac{1}{n}} \rho_X^{-\frac{1}{2n}} \otimes I_Y\right)^n$$

  * Cerf & Adami studied $\rho_{Y|X}^{(\infty)} = \lim_{n\to\infty} \rho_{Y|X}^{(n)}$
  * $\rho_{Y|X}^{(\infty)} = \exp\left(\log \rho_{XY} - (\log \rho_X) \otimes I_Y\right)$ when well-defined.
  * Conditional von Neumann entropy

$$S(Y|X) = S(X,Y) - S(X) = -\mathrm{Tr}\left(\rho_{XY} \log \rho_{Y|X}^{(\infty)}\right)$$

* Many people (including me) have studied $\rho_{Y|X}^{(1)}$ which we'll write $\rho_{Y|X}$
  * Can be characterized as a +ve operator satisfying

$$\mathrm{Tr}_Y\left(\rho_{Y|X}\right) = I_{\mathrm{supp}(\rho_X)}$$

2. Remarks on Conditional Independence

- For $\rho^{(\infty)}_{Y|X}$ the natural definition of conditional independence is entropic:

$$S(X:Y|Z) = 0$$

- Equivalent to equality in Strong Subadditivity:

$$S(X,Z) + S(Y,Z) \geq S(X,Y,Z) + S(Z)$$

- Ruskai showed it is equivalent to

$$\rho^{(\infty)}_{Y|XZ} = \rho^{(\infty)}_{Y|Z} \otimes I_X$$

- Hayden et. al. showed it is equivalent to

$$\rho_{XYZ} = \sum_j p_j \sigma_{X_j Z_j^{(1)}} \otimes \tau_{X_j Z_j^{(2)}} \qquad \mathcal{H}_{XYZ} = \bigoplus_j \mathcal{H}_{X_j Z_j^{(1)}} \otimes \mathcal{H}_{Y_j Z_j^{(1)}}$$

- Surprisingly, it is also equivalent to

$$\rho_{XY|Z} = \rho_{X|Z}\rho_{Y|Z} \qquad \rho_{XYZ} = \rho_{XZ}\rho_Z^{-1}\rho_{YZ}$$

2. Remarks on Conditional Independence

* But this is not the "natural" definition of conditional independence for $n = 1$

  * $\rho_{Y|XZ} = \rho_{Y|Z} \otimes I_X$ and $\rho_{X|YZ} = \rho_{X|Z} \otimes I_Y$ are strictly weaker and inequivalent to each other.

* Open questions:

  * Is there a hierarchy of conditional independence relations for different values of n?

  * Do these have any operational significance in general?

3. Choi-Jamiolkowsi Revisited

* CJ isomorphism is well known. I want to think of it slightly differently, in terms of conditional density operators

$$\rho_{Y|X} \cong \mathcal{E}_{Y|X} : \mathfrak{B}(\mathcal{H}_X) \to \mathfrak{B}(\mathcal{H}_Y)$$

Trace Preserving
Completely Positive
TPCP

* Let $\rho^+_{X'|X} = \sum_{jk} |j\rangle\langle k|_X \otimes |j\rangle\langle k|_{X'}$

* $\mathcal{E}_{Y|X} \to \rho_{Y|X}$ direction

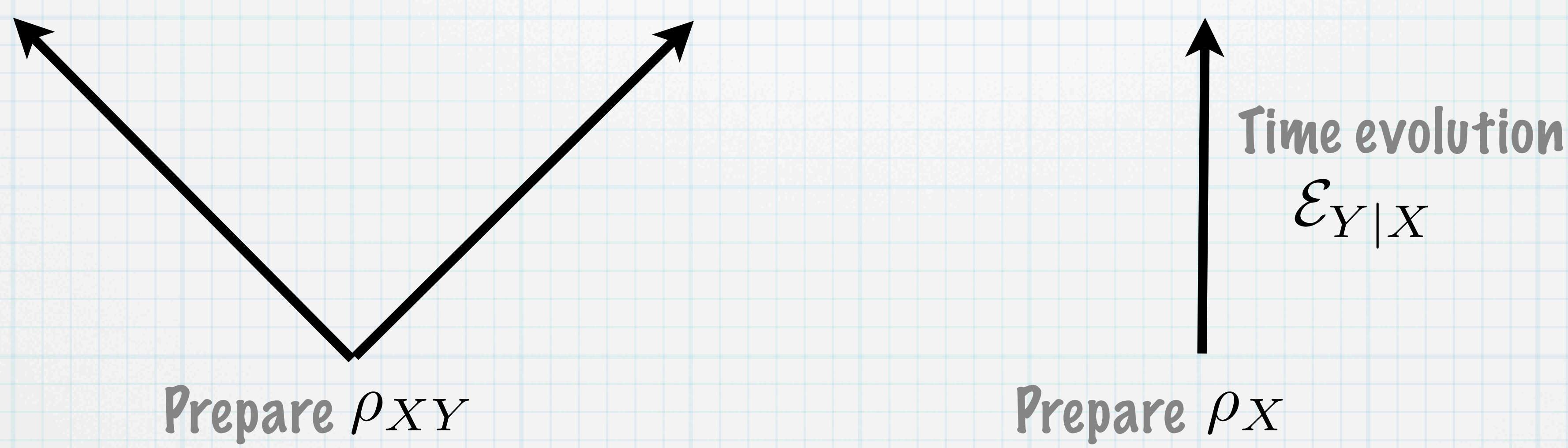
$$\rho_{Y|X} = \mathcal{I}_X \otimes \mathcal{E}_{Y|X'}\left(\rho^+_{X'|X}\right)$$

* $\rho_{Y|X} \to \mathcal{E}_{Y|X}$ direction

$$\mathcal{E}_{Y|X}(\sigma_X) = \mathrm{Tr}_{XX'}\left(\rho^+_{X'|X}\sigma_X \otimes \rho_{Y|X'}\right)$$

3. Choi-Jamiolkowsi Revisited

* What does it all mean? $\rho_{XY} \cong \left(\rho_X, \rho_{Y|X}\right) \cong \left(\rho_X, \mathcal{E}_{Y|X}\right)$
* Cases 2. and 3. from the intro are already unified in QM, i.e. both experiments



* can be described simply by specifying a joint state ρ_{XY}.
* Do expressions like $\mathrm{Tr}\left(M_X \otimes N_Y \rho_{XY}\right)$, where M_X, N_Y are POVM elements have any meaning in the time-evolution case?

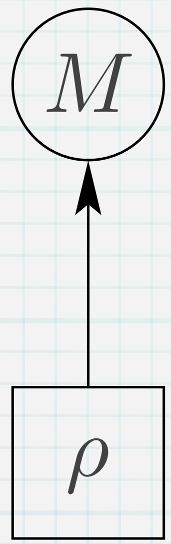
3. Choi-Jamiolkowski Revisited

* Lemma: $\rho = \sum_j p_j \rho_j$ is an ensemble decomposition of a density matrix ρ iff there is a POVM $M = \left\{ M^{(j)} \right\}$ s.t.

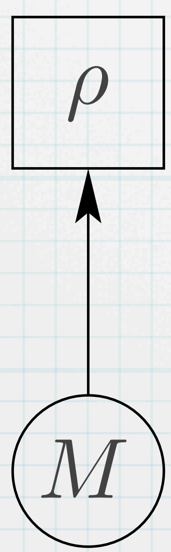
$$p_j = \mathrm{Tr}\left(M^{(j)} \rho \right) \quad \text{and} \quad \rho_j = \frac{\sqrt{\rho} M^{(j)} \sqrt{\rho}}{\mathrm{Tr}\left(M^{(j)} \rho \right)}$$

* Proof sketch: Set $M^{(j)} = p_j \rho^{-\frac{1}{2}} \rho_j \rho^{-\frac{1}{2}}$

3. Choi-Jamiolkowski Revisited



* M-measurement of ρ
    * Input: ρ
    * Measurement probabilities: $P(M = j) = \text{Tr}\left(M^{(j)}\rho\right)$



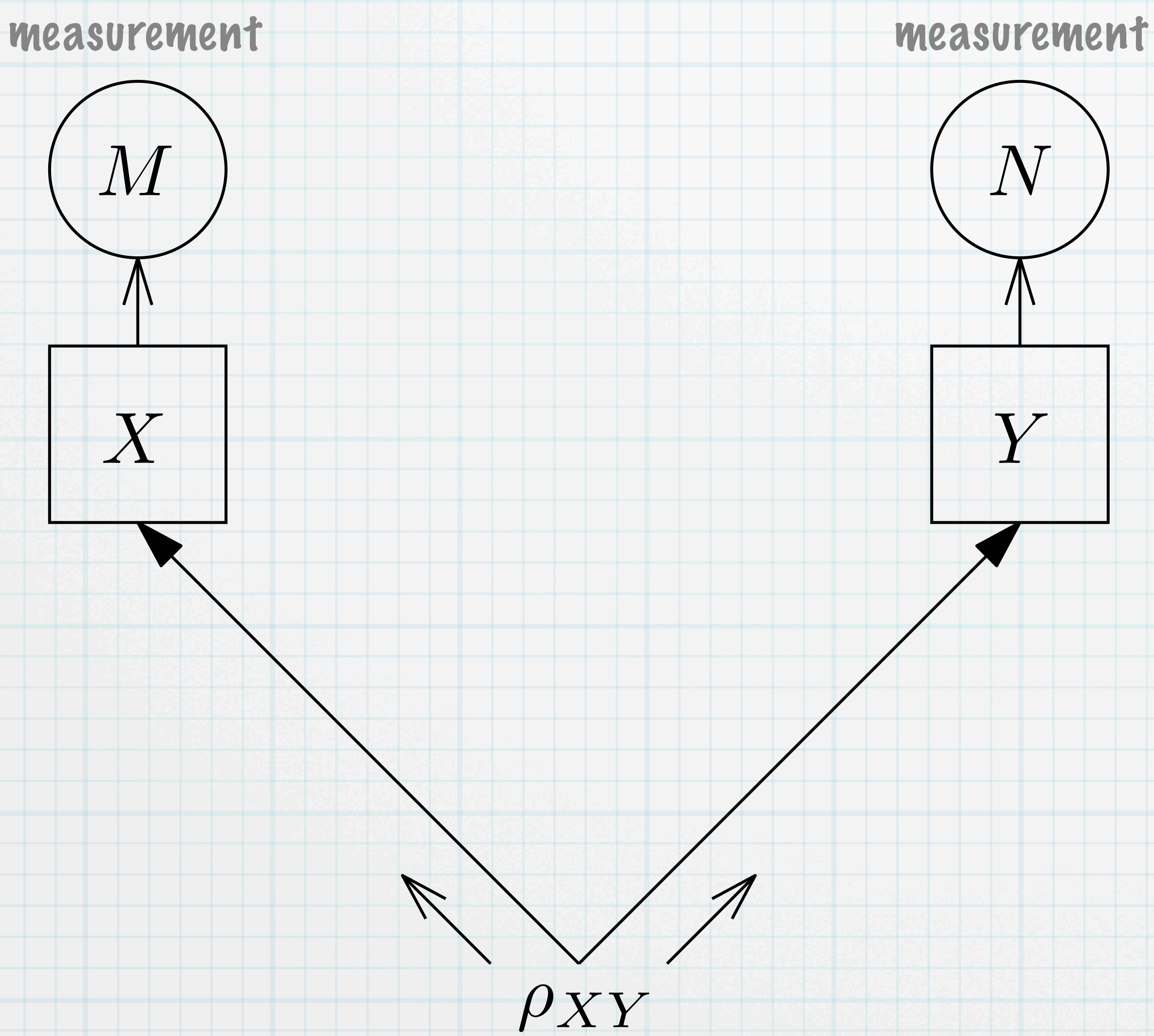
* M-preparation of ρ
    * Input: Generate a classical r.v. with p.d.f

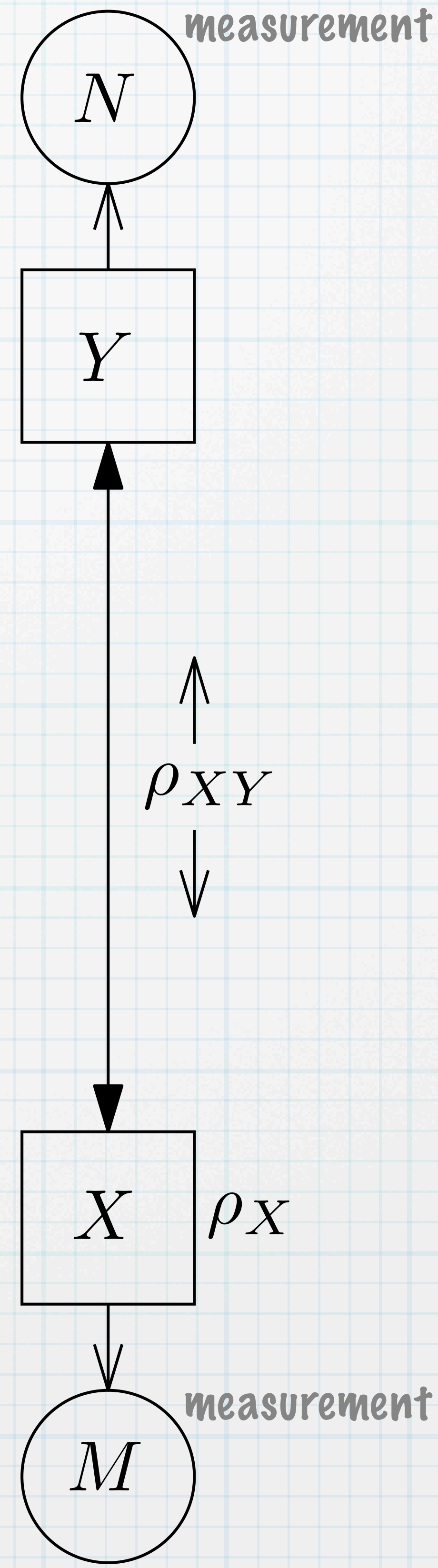
$$P(M = j) = \text{Tr}\left(M^{(j)}\rho\right)$$

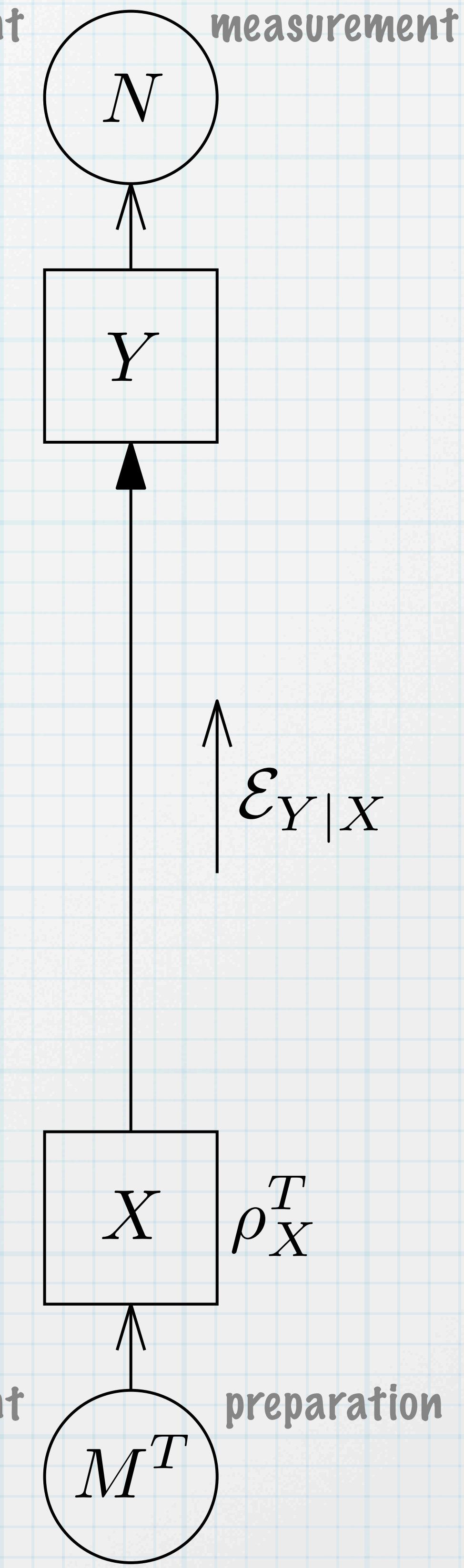
* Prepare the corresponding state:

$$\rho_j = \frac{\sqrt{\rho}M^{(j)}\sqrt{\rho}}{\text{Tr}\left(M^{(j)}\rho\right)}$$

3. Choi-Jamiolkowski revisited







$P(M, N)$ is the same for any POVMs M and N.

4. Applications

* Relations between different concepts/protocols in quantum information, e.g. broadcasting and monogamy of entanglement.
* Simplified definition of quantum sufficient statistics.
* Quantum State Pooling.
* Re-examination of quantum generalizations of Markov chains, Bayesian Networks, etc.

5. Open Questions

* Is there a hierarchy of conditional independence relations?
* Operational meaning of conditional density operator and conditional independence for general n?
* Temporal joint measurements, i.e. $\mathrm{Tr}\left(M_{XY}\rho_{XY}\right)$?
* The general quantum conditional probability question.
* Further applications in quantum information?